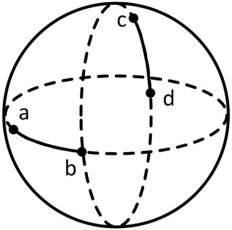
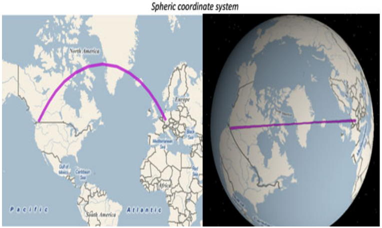
4. Spherical Geometry

A First Look at Spherical Geometry

Great Circle: A circle on the sphere with center at the center of the sphere. Think of a plane passing through the centre of the sphere - it intersects the sphere in a great circle.

The shortest path between two points follows a great circle.





Great circles

A **great circle** on a sphere is the intersection of this sphere with a plane passing through its centre.

If the earth were a sphere then the equator would be a great circle.

Two points on a sphere are termed **antipodal** if they are the intersection of the sphere with a straight line that passes through its centre – i.e. they lie on opposite ends of a diameter of the sphere.

Again if the earth were a sphere then the north and south poles would be antipodal points.

Observe that three points, not collinear, determine a unique plane in $\mathbb{R}^3$.

Theorem

Let S be a sphere in $\mathbb{R}^3$. Let a, b be any two distinct points of S that are not antipodal. Then there is a unique great circle that contains a, b.

Proof: Let o denote the centre of the sphere. As a, b are not antipodal points (i.e. they are not the endpoints of a diameter of the sphere), a, b, o are not collinear. Hence a, b, o determine a unique plane in $\mathbb{R}^3$. The intersection of this unique plane with the sphere S, determines a unique great circle of S.

Spherical triangles

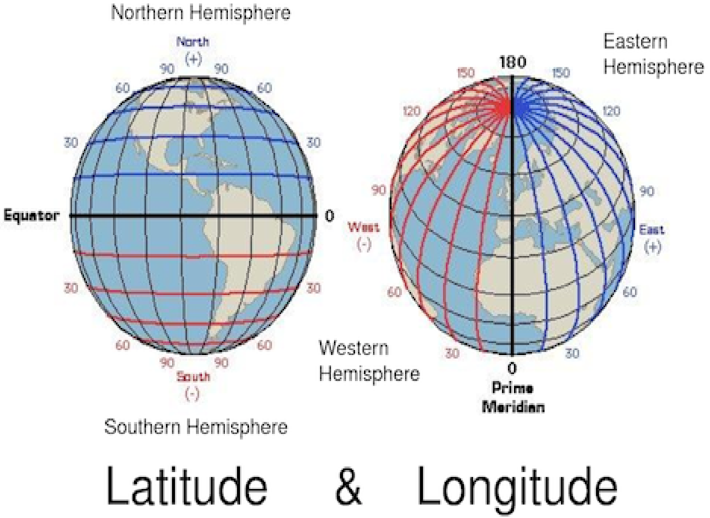
In spherical geometry, the lines are the great circles.

There are no parallel lines.

Any two distinct great circles intersect in two points, which are antipodal.

A **spherical triangle** consists of three points a, b, c on a sphere, not collinear (not on the same great circle), and they are joined by the three distinct lines (great circles) containing the pairs (a, b), (a, c) and (b, c).

Coordinates on the Sphere: Latitude & Longitude



Latitude: Angle up or down from the Equator.
Varies between $90°$ S and $90°$ N (i.e., $-90°$ and $90°$.)
The points with the same latitude form a **parallel of latitude**.

Longitude: Angle measured east or west of Greenwich.
Varies between $180°$ W and $180°$ E ($-180°$ and $180°$.)
Points with the same longitude lie on a **meridian of longitude**.

Galway: $53.270962°$ N, $9.062691°$ W.
Sometimes given in **degrees, minutes, seconds:**
$53° \ 16' \ 15''$ N, $9° \ 3' \ 46''$ W.
(1 degree = 60 minutes, 1 minute = 60 seconds.)

Spheres in $\mathbb{R}^3$

We view the Earth as a sphere in $\mathbb{R}^3$ (even if it really isn't).

The equation of a sphere with centre at the origin and radius r:

$$x^2 + y^2 + z^2 = r^2$$

The simplest case is the unit sphere:

$$x^2 + y^2 + z^2 = 1$$

The radius of the Earth is approximately 6,378 kilometres. We can choose our unit of measurement so that this becomes the unit sphere. In other words, the earth will have radius $r = 1$.

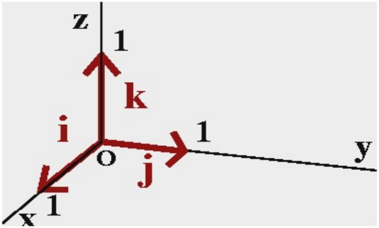
Vectors in $\mathbb{R}^3$

Vectors:

$$\mathbf{X} = (x_1, x_2, x_3); \mathbf{Y} = (y_1, y_2, y_3), \text{ etc.}$$

The unit vectors along the coordinate axes are usually denoted

$$\mathbf{i} = (1, 0, 0); \mathbf{j} = (0, 1, 0); \mathbf{k} = (0, 0, 1)$$



These are used to decompose vectors:

$$\mathbf{X} = (1, -2, 4) = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}; \quad \mathbf{Y} = (0, 1, 3) = \mathbf{j} + 3\mathbf{k}$$

$$\mathbf{X} = (1, -2, 4) = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}; \quad \mathbf{Y} = (0, 1, 3) = \mathbf{j} + 3\mathbf{k}$$

To add vectors, we add their components:

$$\mathbf{X} + \mathbf{Y} = (1, -1, 7) = \mathbf{i} - \mathbf{j} + 7\mathbf{k}$$

And we can multiply a vector by a scalar (real number):

$$5\mathbf{X} = (5, -10, 20) = 5\mathbf{i} - 10\mathbf{j} + 20\mathbf{k}$$

Two vectors are **parallel** if one is a scalar multiple of the other:

$$(1, -1, 7) \parallel (-2, 2, -14).$$

The Dot Product (Scalar Product)

$$\mathbf{X} \cdot \mathbf{Y} = x_1 y_1 + x_2 y_2 + x_3 y_3$$

So, for the vectors $\mathbf{X} = (1, -2, 4)$, $\mathbf{Y} = (0, 1, 3)$, we have

$$\mathbf{X} \cdot \mathbf{Y} = (1)(0) + (-2)(1) + (4)(3) = 10$$

Some properties of dot products:

1. $\mathbf{X} \cdot \mathbf{Y} = \mathbf{Y} \cdot \mathbf{X}$
2. $\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$
3. $\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$
4. Length: $|\mathbf{X}| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{\mathbf{X} \cdot \mathbf{X}}$

For $X = (1, -2, 4)$, we get $|\mathbf{X}| = \sqrt{21}$.

Geometric Interpretation of the Dot Product

Let θ be the angle between the vectors $\mathbf{X}$ and $\mathbf{Y}$ that is in the range $0 \leqslant \theta < 180°$. Then

$$\mathbf{X} \cdot \mathbf{Y} = |\mathbf{X}||\mathbf{Y}| \cos(\theta)$$

We can use this to find the angle between two vectors:

$$\theta = \cos^{-1}\left(\frac{\mathbf{X} \cdot \mathbf{Y}}{|\mathbf{X}||\mathbf{Y}|}\right).$$

Two (non-zero) vectors $\mathbf{X}$, $\mathbf{Y}$ are **orthogonal (perpendicular)** if and only if their dot product is zero:

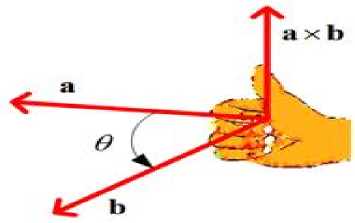
$$\mathbf{X} \perp \mathbf{Y} \iff \mathbf{X} \cdot \mathbf{Y} = 0.$$

The Cross Product (Vector Product)

The cross product $\mathbf{X} \times \mathbf{Y}$ gives a vector that is orthogonal to both $\mathbf{X}$ and $\mathbf{Y}$.

- The direction is chosen so that $\mathbf{X} \times \mathbf{Y}$ is orthogonal to the plane containing $\mathbf{X}$ and $\mathbf{Y}$ and so that the three vectors $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{X} \times \mathbf{Y}$ form a 'right-handed system'.
- The length of $\mathbf{X} \times \mathbf{Y}$ is given by

$$|\mathbf{X} \times \mathbf{Y}| = |\mathbf{X}||\mathbf{Y}| \sin(\theta).$$



Basic properties of cross products

1. $\mathbf{Y} \times \mathbf{X} = -\mathbf{X} \times \mathbf{Y}$.
2. $\mathbf{X} \times \mathbf{X} = 0$.
3. $\mathbf{i} \times \mathbf{j} = \mathbf{k},\ \mathbf{j} \times \mathbf{k} = \mathbf{i},\ \mathbf{k} \times \mathbf{i} = \mathbf{j}$.

For example, let $\mathbf{X} = (1, 2, -4)$ and $\mathbf{Y} = (0, 1, 3)$. Then

$$\begin{aligned}
\mathbf{X} \times \mathbf{Y} &= (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) \times (\mathbf{j} + 3\mathbf{k}) \\
&= \mathbf{i} \times \mathbf{j} + 3(\mathbf{i} \times \mathbf{k}) - 2(\mathbf{j} \times \mathbf{j}) - 6(\mathbf{j} \times \mathbf{k}) + 4(\mathbf{k} \times \mathbf{j}) + 12(\mathbf{k} \times \mathbf{k}) \\
&= \mathbf{k} + 3(-\mathbf{j}) - 2(0) - 6\mathbf{i} + 4(-\mathbf{i}) + 12(0) \\
&= -10\mathbf{i} - 3\mathbf{j} + \mathbf{k} = (-10, -3, 1).
\end{aligned}$$