Class test 1: Thursday 28 Feb, AM150

- Problem sheet parts I - III.
- Lectures 1-11.
- Short questions, may include theorems, proofs, examples, calculations.
- Bring paper to write on and a calculator.

Projective Transformations

Every projective transformation is of the form

$$f(x) = \frac{ax + b}{cx + d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc \neq 0$.

Each transformation can be decomposed into the composition of functions, each of which is one of the following:

1. Translation: $f_1(x) = x + a$;
2. Scaling: $f_2(x) = cx$;
3. Inversion: $f_3(x) = 1/x$.

The Cross Ratio

This is a number that measures the relative spacing between a set of four points on a line.

Let A, B, C, D be four points on a line. Their Cross Ratio is defined by

$$[A, B, C, D] = \frac{|AC||BD|}{|BC||AD|}.$$

If one of the four points A, B, C, D is the point at infinity, then the cross ratio is defined by omitting the distances that refer to this point. For example, if A is the point at infinity, then the cross ratio is defined to be

$$\frac{\cancel{|AC|}|BD|}{|BC|\cancel{|AD|}} = \frac{|BD|}{|BC|}.$$

Theorem

The cross ratio of four points on a line is preserved by every projective transformation.

Proof of Theorem:

Since every projective transformation is a composition of translations, scalings and inversions, we just have to show that each of these preserves the cross ratio. 

Corollary (The Secret of Perspective Drawing)

In a perspective drawing of four equally spaced points, the cross ratio must be 4/3.

Proof: The cross ratio of four equally spaced points is 4/3, so any projection must preserve this number.

Example: Suppose you are making a perspective drawing that includes three posts in a line. They are 4m, 3m and 1m respectively from a wall. You have placed the two farthest posts 3cm and 1cm from the wall in your drawing. Where should you place the third post? 

The car example

An aerial camera is showing a car travelling along a straight road towards a junction. There are warning signs 4 km and 2 km from this junction. On the screen, these signs are 1 cm and 3 cm from the junction and the car is 3/7 cm from the junction. How far, on the ground, is the car from the junction? 

Another example

An aerial photograph shows a train travelling between two stations on a straight track. The stations are 50 km apart. On the film, the stations are 4 cm apart, the train has covered three quarters of the distance between the stations and the tracks appear to meet 4 cm beyond the second station. How far is the train away from the second station?