Projective Transformations

When we project from one plane to another, very few features are preserved. Points are still points, lines are still lines and crossings between lines are preserved.

But angles and distances change. The relative spacing between three points on a line also changes.

What features are preserved in a perspective view?

The spacing between a pair of points can be changed to any value. It is **not** invariant.

The same is true of the relative spacing between any three points on a line: it is **not** invariant.

However, there is a number that describes the relative spacing between four points on a line that **is** invariant.

The Cross Ratio

This is a number that measures the relative spacing between a set of four points on a line.

Let A, B, C, D be four points on a line. Their Cross Ratio is defined by

$$[A, B, C, D] = \frac{|AC||BD|}{|BC||AD|}.$$

Example: Find the cross ratio between four equally spaced points.

Dealing with the point at infinity

A projective transformation may send one of the four points to the point at infinity, so we have to be able to deal with this situation.

If one of the four points $A, \mathrm{B}, \mathrm{C}, \mathrm{D}$ is the point at infinity, then the cross ratio is defined by omitting the distances that refer to this point.

For example, if A is the point at infinity, then the cross ratio is defined to be

$$\frac{\cancel{|AC|}|\mathrm{BD}|}{|\mathrm{BC}|\cancel{|AD|}} = \frac{|\mathrm{BD}|}{|\mathrm{BC}|}.$$

Rationale

Picture what happens when A moves far away from $\mathsf{B}, \mathsf{C}, \mathsf{D}$.

$$\frac{|AC||BD|}{|BC||AD|} = \frac{|AC|}{|AD|} \cdot \frac{|BD|}{|BC|}.$$

where as $A \to \infty$, $\frac{|AC|}{|AD|} \to 1$.

We will show that the cross ratio for four points on a line is preserved by every projective transformation.

Recall the three basic types of projective transformation

1. **Scaling**

$$x \to ax$$

2. **Inversion**

$$x \to \frac{1}{x}$$

This mapping sends 0 on one line to 1 on the other and vice versa.

3. **Translation**

$$x \to x + b$$

If we combine any number of these in any order, we end up with a function like this:

$$f(x) = \frac{ax + b}{cx + d}$$

A function of the form

$$f(x) = \frac{ax+b}{cx+d}$$

is called a **Projective Transformation**.

A projective transformation projects one line onto another. Two points to note:

1. When working with projective transformations, we take $\frac{1}{0}$ to mean ∞, the point at infinity.
2. We have to add the condition $ad - bc \neq 0$ in order to be able to invert a projective transformation.

If $ad - bc = 0$, then every point is mapped to the same point (the function is constant.)

Theorem

Every Projective Transformation is a composition of the three basic types: Scaling, Inversion, Translation.

For example let $f(x) = \frac{2x+1}{x-2}$.

Look at the division:

$$f(x) = \frac{2x+1}{x-2} = 2 + \frac{5}{x-2}.$$

So $f(x)$ can be got from the following sequence:

1. Translate x by -2 to get $x-2$;
2. Invert to get $1/(x-2)$;
3. Scale by 5 to get $5/(x-2)$;
4. Translate by 2 to get $2+5/(x-2)$.

Proof:

$$f(x) = \frac{ax+b}{cx+d}$$

If $c = 0$, then $f(x) = (a/d)x + (b/d)$: scaling followed by translation.

If $c \neq 0$, divide $ax + b$ by $cx + d$ to get an expression of the form

$$f(x) = \alpha + \frac{\beta}{cx+d}$$

so $f(x)$ can be decomposed into scaling[1], translation[2], inversion[3], scaling[4], translation[5].

$$x \xrightarrow{1} cx \xrightarrow{2} cx + d \xrightarrow{3} \frac{1}{cx+d} \xrightarrow{4} \frac{\beta}{cx+d} \xrightarrow{5} \alpha + \frac{\beta}{cx+d}$$

Another example

$$f(x) = \frac{x-5}{2x+3}$$

The cross ratio

Theorem

The cross ratio is preserved by every projective transformation.

Corollary (The Secret of Perspective Drawing)

In a perspective drawing of four equally spaced points, the cross ratio must be 4/3.

Proof of Theorem:

Since every projective transformation is a composition of translations, scalings and inversions, we just have to show that each of these preserves the cross ratio.