

lec
9

Polar Form of Complex

Numbers — Powers and Roots.

We can substantially increase the usefulness of the complex plane

if besides the x, y coordinates we also employ the usual polar coordinates r, θ defined by

$$\left. \begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \right\} *$$

So, $z = x + iy$
takes polar form

$$\begin{aligned} z &= r \cos \theta + i r \sin \theta \\ &= r (\cos \theta + i \sin \theta) \end{aligned}$$

r here is the absolute value or modulus of z
(we had) $= |z|$

$$\text{So } |z| = r = \sqrt{x^2 + y^2} = \sqrt{z \bar{z}}$$

$\leftarrow \text{new} \rightarrow$

We said $|z|$ (or r) is the distance from z to the origin.

(Similarly $|z_1 - z_2|$ is the distance between z_1 and z_2 .)

θ is called the argument of z .

It is denoted by $\arg(z)$.

Dividing the 2nd equation by the 1st, we get

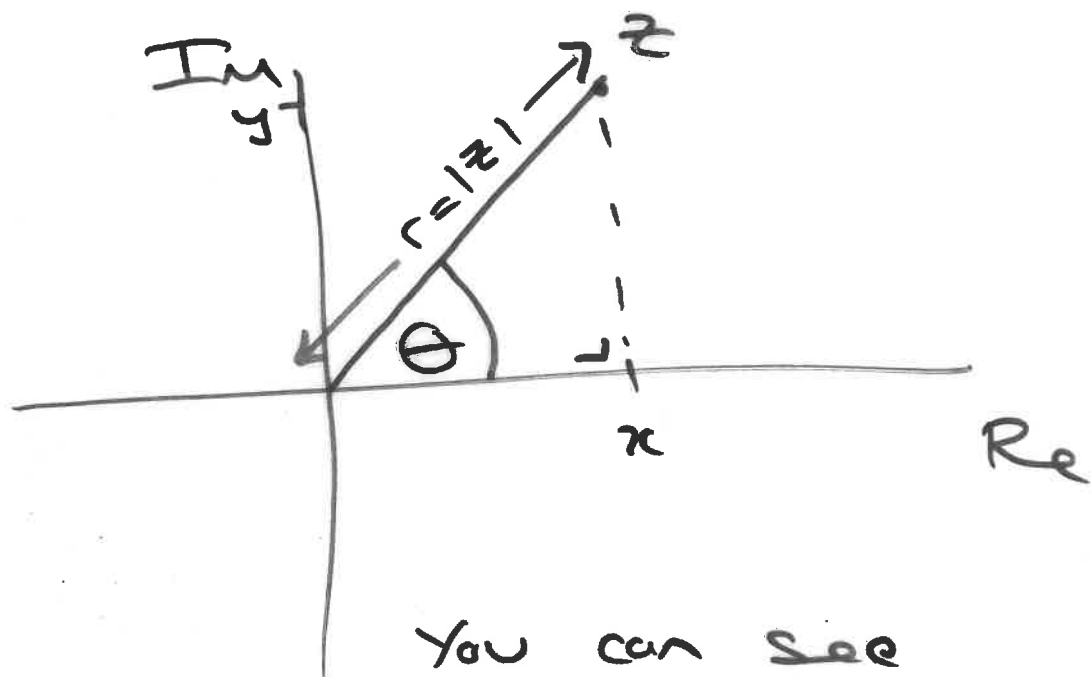
$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta}$$

$$\text{or } \frac{y}{x} = \frac{\sin \theta}{\cos \theta}$$

$$\text{or } \frac{y}{x} = \tan \theta \quad \left[\tan \theta = \frac{y}{x} \right]$$

$$\text{or } \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

Geometrically, we have



You can see

$$\tan \theta = \frac{\text{OPP}}{\text{ADJ}} = \frac{y}{x}$$

All angles are measured in radian and positive in the counterclockwise sense.

Ex $z = 1 + i$

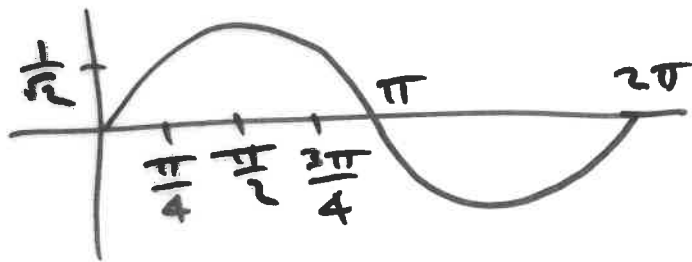
First: find r .

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

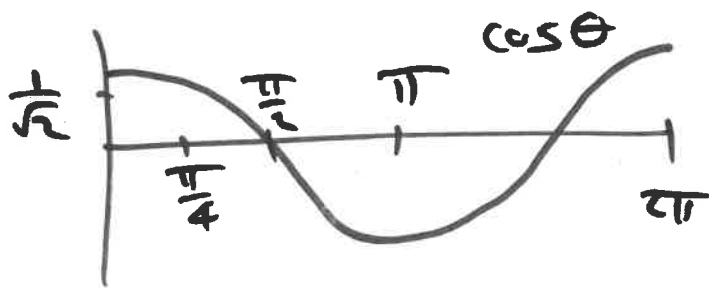
$$\text{So } z = \sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$$

This means $\frac{1}{\sqrt{2}}$ is $\cos \theta$ and $\frac{1}{\sqrt{2}}$ is $\sin \theta$.

$\sin \theta$



$\theta = \frac{\pi}{4}$ or $\frac{3\pi}{4}$
appears to work



$\theta = \frac{\pi}{4}$ or $\frac{7\pi}{4}$
appears to work

$\theta = \frac{\pi}{4}$ is the only value (between 0 and 2π) that satisfies both requirements.

Writing $z = 1 + i$ in polar coordinates, we have

$$z = \frac{1}{\sqrt{2}} \left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right)$$

Note, in the previous example we can add or subtract any integer multiple of 2π to θ .

This is because the functions $\sin \theta$ and $\cos \theta$ are periodic in 2π .

e.g. (from the last example)

$$z = \sqrt{2} \left[\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right]$$

equals

$$\sqrt{2} \left[\cos\left(\frac{\pi}{4} + n(2\pi)\right) + i \sin\left(\frac{\pi}{4} + n(2\pi)\right) \right]$$

$n \in \mathbb{Z}$

Definition

The principal value of the argument of z ($\neq 0$)

is denoted $\text{Arg}(z)$, with capital 'A'

$\theta = \text{Arg}(z)$ satisfies:

$$-\pi < \text{Arg}(z) \leq \pi$$

[Practical tip:

When calculating $\text{Arg}(z)$,
find θ such that $0 \leq \theta < 2\pi$
(as in the previous example),

If $\theta > \pi$, just subtract 2π
from your answer.]

Ex

Express $z = -3 - 3\sqrt{3}i$

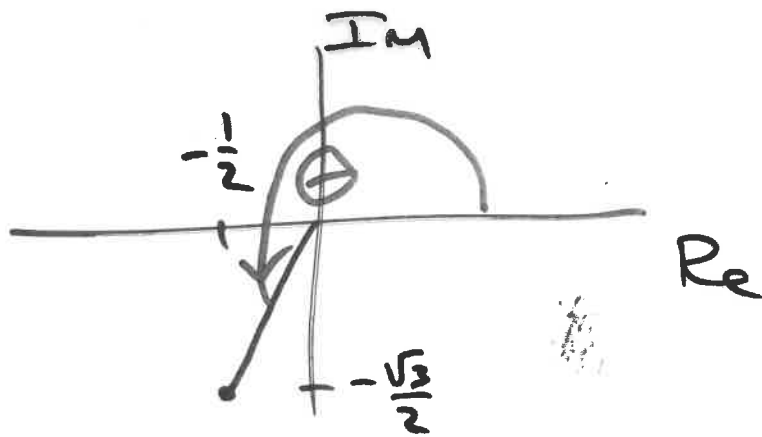
(a) in polar form

(b) Find $\text{Arg}(z)$.

Sol

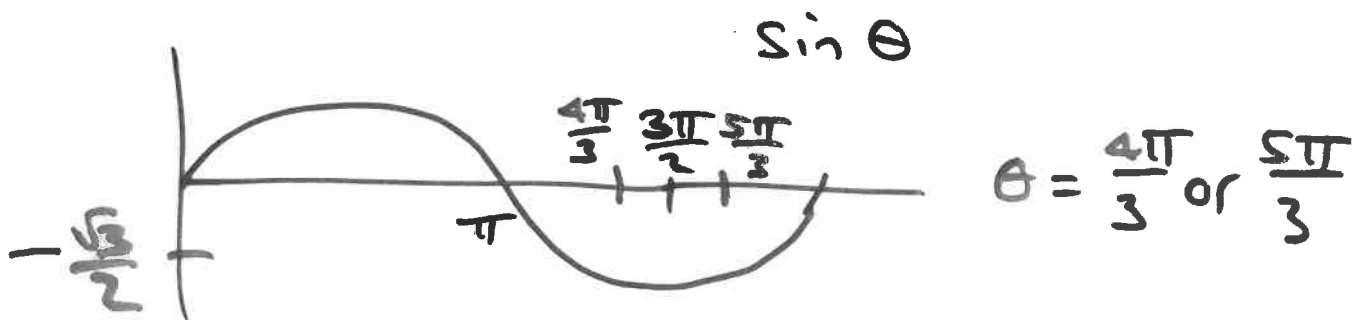
$$\begin{aligned} \text{(a)} \quad r &= |z| = \sqrt{(-3)^2 + (-3\sqrt{3})^2} \\ &= \sqrt{9 + 27} = \sqrt{36} = +6 \end{aligned}$$

$$\begin{aligned} \Rightarrow z &= 6 \left(\underbrace{-\frac{1}{2}}_{\cos \theta} - \underbrace{\frac{\sqrt{3}}{2}i}_{i \sin \theta} \right) \\ &= 6 \left(\cos \theta + i \sin \theta \right) \end{aligned}$$

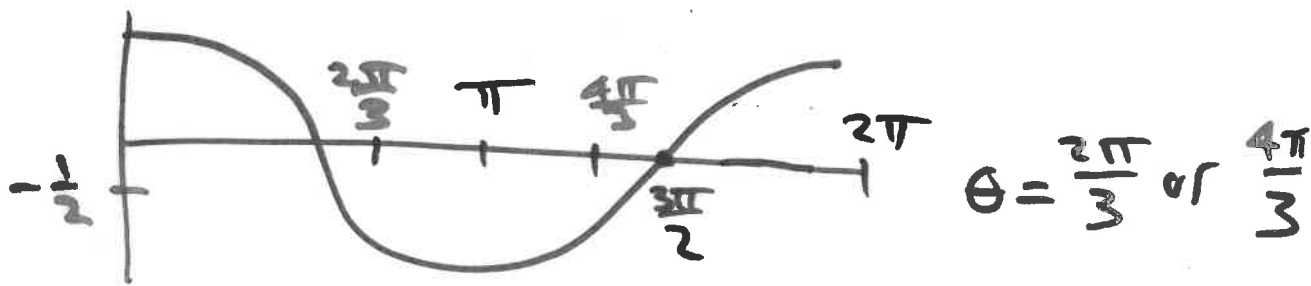


From the diagram $\theta = \frac{4\pi}{3}$

OR



$$\theta = \frac{4\pi}{3} \text{ or } \frac{5\pi}{3}$$



$$\theta = \frac{2\pi}{3} \text{ or } \frac{4\pi}{3}$$

The angle that satisfies both requirements is: $\theta = \frac{4\pi}{3}$

So in polar form

$$z = 6 \left(\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) \right)$$

(b) As $-\pi < \text{Arg}(z) \leq \pi$

$$\text{Arg}(z) \neq \frac{4\pi}{3} > \pi.$$

$$\text{Arg}(z) = \frac{4\pi}{3} - 2\pi = -\frac{2\pi}{3}$$

Cautions

When using $\text{Arg}(z) = \text{TAN}^{-1} \frac{y}{x}$,
one has to be careful!

e.g. if $z_1 = 1+i$, $z_2 = -1-i$

$$\text{arg}(z_1) = \text{TAN}^{-1} \frac{1}{1} = \text{TAN}^{-1} 1 = \frac{\pi}{4}$$

$$\text{arg}(z_2) = \text{TAN}^{-1} \frac{\text{Im}}{\text{Re}} = \text{TAN}^{-1} \frac{-1}{-1} = \text{TAN}^{-1} 1 = \frac{\pi}{4} \text{ wrong}$$

However,



Therefore,

when using the formula

$$\text{arg}(z) = \text{TAN}^{-1} \frac{y}{x} \text{ and if } x \text{ or } y < 0$$

justify your answer geometrically

For any complex numbers

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

This is known as the triangle inequality.

The generalized triangle inequality

$$\text{is: } |z_1 + z_2 + \dots + z_n| \leq |z_1| + |z_2| + \dots + |z_n|$$

Ex
If $z_1 = 1 + i$

$$z_2 = -2 + 3i$$

$$|z_1 + z_2| = |-1 + 4i| = \sqrt{(-1)^2 + 4^2} = \sqrt{17}$$

$$|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$|z_2| = \sqrt{(-2)^2 + 3^2} = \sqrt{13}$$

$$\sqrt{17} < \sqrt{2} + \sqrt{13} \quad \checkmark$$