Lec $\boxed{8}$

- If two harmonic functions $u(x,y)$ and $v(x,y)$ satisfy the C-R equations in a domain D, they are the real and imaginary parts of an analytic function f in D.

[$f(z) = u(x,y) + i\, v(x,y)$]

- $v(x,y)$ is said to be the conjugate harmonic function of u in D

(This has nothing to do with the "conjugate" $\overline{z}$)

Ex Earlier we showed

$$u(x,y) = 2x + y + e^x \sin(y)$$

is harmonic.

Find a conjugate harmonic function v of u.

Sol
$$\frac{\partial u}{\partial x} = 2 + e^x \sin y$$

As $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$

we can say: $\dfrac{\partial v}{\partial y} = 2 + e^x \sin y$

or $v(x,y) = 2y - e^x \cos y + h(x)$ *

$$\frac{\partial u}{\partial y} = 1 + e^x \cos y$$

As $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

we can say $\dfrac{\partial v}{\partial x} = -1 - e^x \cos y$

or $v(x,y) = -x - e^x \cos y + \ell(y)$

Rewriting * $v(x,y) = h(x) - e^x \cos y + 2y$

Therefore, $h(x) = -x$

and $\ell(y) = 2y$

So $v(x,y) = -x - e^x \cos y + 2y$

is a conjugate harmonic function of $u(x,y)$.

Note: we could have picked

$h(x) = -x + c_1$ or

$\ell(y) = 2y + c_2$ $c_1, c_2 \in \mathbb{R}$

So, there are infinitely many conjugate harmonic functions of $u(x,y)$, given by

$v(x,y) = -x - e^x \cos y + 2y + C$