Lec $\boxed{7}$

Laplace's equation

If $f(z) = u(x,y) + i\,v(x,y)$

is analytic in a domain D,

then u and v satisfy

Laplace's equation

i.e. $\nabla^2 u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ①

and $\nabla^2 v = \dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ ②

[Notation,

we write $\dfrac{\partial u}{\partial x} = u_x$, $\dfrac{\partial^2 u}{\partial y^2} = u_{yy}$

etc.

Proof of ① Differentiating :

$u_x = V_y$ with respect to x

$u_y = -V_x$ with respect to y

we get :

$u_{xx} = V_{yx}$

and $u_{yy} = -V_{xy}$, respectively.

Adding the above two equations

we get $u_{xx} + u_{yy} = 0$

Proof of ② Differentiating :

$u_x = V_y$ with respect to y

$u_y = -V_x$ with respect to x

we get :

$u_{xy} = V_{yy}$

$u_{xy} = -V_{xx}$

Subtracting the above equations we get

$$0 = V_{yy} + V_{xx}.$$

i.e. $V_{xx} + V_{yy} = 0$

Any function that satisfies Laplace's equation is said to be a <u>harmonic function</u>

Note: we proved that if

$f(z) = u(x,y) + i\,v(x,y)$

is analytic,

then $u(x,y)$ and $v(x,y)$ are harmonic functions.

<u>Ex</u>

Show that

$u(x,y) = 2x + y + e^{x}\sin y$ is harmonic

Sol

$u_x = 2 + e^x \sin y$

$u_{xx} = e^x \sin y$

$u_y = 1 + e^x \cos y$

$u_{yy} = -e^x \sin y$

$u_{xx} + u_{yy} = 0$ ✓