

Leq 6
Ex Is $f(z) = z^3$ analytic?

$$\begin{aligned} z^3 &= (x + iy)^3 \\ &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 \\ &= x^3 + (3x^2y)i - 3xy^2 - (y^3)i \\ &= [x^3 - 3xy^2] + [3x^2y - y^3]i \end{aligned}$$

As $f(z) = u(x, y) + i v(x, y)$

$$u(x, y) = x^3 - 3xy^2$$

$$v(x, y) = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -3x(2y) = -6xy$$

$$\frac{\partial v}{\partial x} = 6xy$$

$$\therefore \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The Cauchy - Riemann equations are satisfied $\forall z \in \mathbb{C}$
Therefore, $f(z)$ is analytic over all \mathbb{C}

Ex

Use the Cauchy - Riemann equations

to find all the points

$z = x + iy$ where

$$f(z) = x^2 - 2xy - y + i(x + x^2 + y^2)$$

is differentiable

Sol

$$u(x, y) = x^2 - 2xy - y$$
$$v(x, y) = x + x^2 + y^2$$

$$\frac{\partial u}{\partial x} = 2x - 2y$$

$$\frac{\partial v}{\partial y} = 2y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\text{or } 2x - 2y = 2y$$

$$\Rightarrow 2x = 4y$$

$$\Rightarrow y = \frac{1}{2}x$$

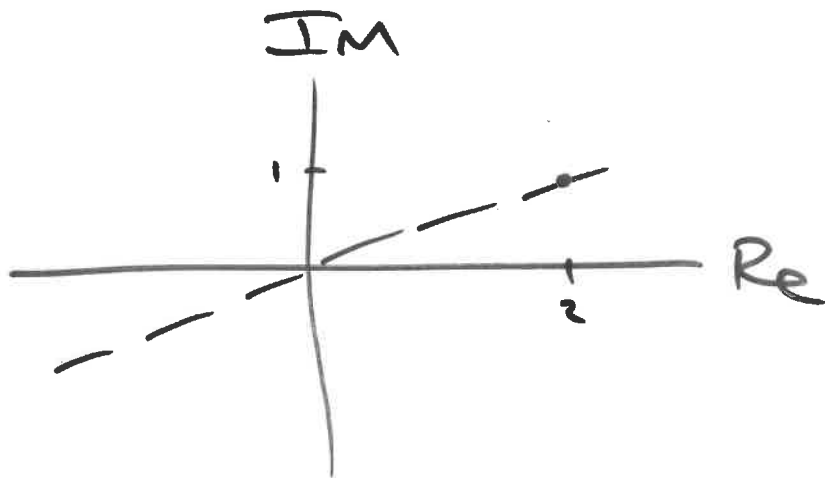
The other C-R equation,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial u}{\partial y} = -2x - 1 \quad , \quad \frac{\partial v}{\partial x} = 1 + 2x$$

$\Rightarrow \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ is satisfied

Therefore, $f(z)$ is differentiable
along the line $y = \frac{1}{2}x$
in the complex plane.



Question:

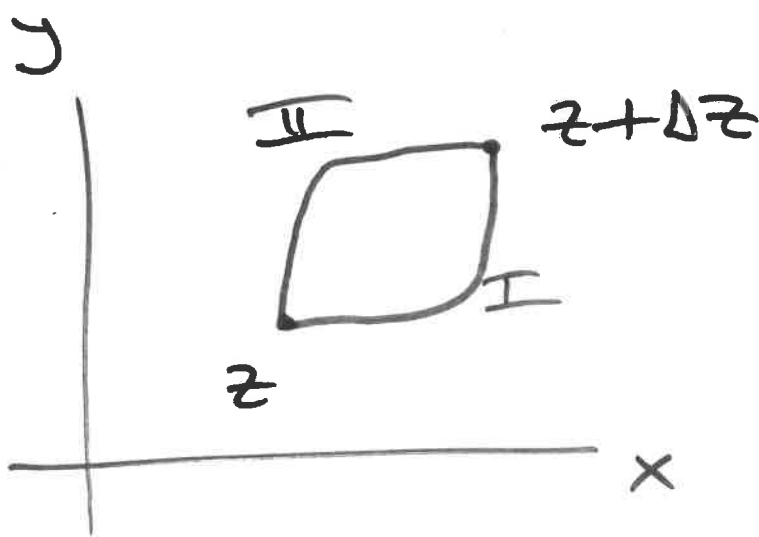
How do we compute $f'(z)$
where it is differentiable?

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} *$$

In the case of a real variable (Δx)

it can only go to zero in two ways.

Δz can go to zero many ways i.p.



Writing * out in full, we get

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{[u(x + \Delta x, y + \Delta y) + i v(x + \Delta x, y + \Delta y)] - [u(x, y) + i v(x, y)]}{\Delta x + i \Delta y}$$

Moving along path I

i.e. Let $\Delta y \rightarrow 0$ first ($\Delta y = 0$)
and then $\Delta x \rightarrow 0$.

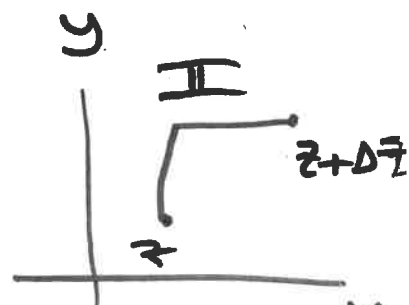
$$f'(z) = \lim_{\Delta x \rightarrow 0} \frac{u(x+\Delta x, y) - u(x, y)}{\Delta x} +$$

$$i \lim_{\Delta x \rightarrow 0} \frac{v(x+\Delta x, y) - v(x, y)}{\Delta x}$$

Note: the above limits are partial derivatives

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad (1)$$

Moving along path II



i.e. Let $\Delta x \rightarrow 0$ first ($\Delta x = 0$)
and then $\Delta y \rightarrow 0$

$$f'(z) = \lim_{\Delta y \rightarrow 0} \frac{u(x, y+\Delta y) - u(x, y)}{i\Delta y}$$

$$+ i \lim_{\Delta y \rightarrow 0} \frac{v(x, y+\Delta y) - v(x, y)}{i\Delta y}$$

$$\begin{aligned}
 \text{or } f'(z) &= \frac{1}{i} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\
 &= -i \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \\
 &= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial x} \quad (2)
 \end{aligned}$$

Equating the real and imaginary parts of (1) and (2) we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

i.e. the C-R equations.

One could use (1) or (2) to compute $f'(z)$.

Ex Find $f'(z)$ for the previous example.

$$[f(z) = x^2 - 2xy - y + i(x + x^2 + y^2)]$$

Using (1) i.e. $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$$\text{So } f'(z) = [2x - 2y] + i [1 + 2x]$$

Recall: The derivative only exists along $y = \frac{1}{2}x$ in this \mathbb{E}

Ex (a) At what points (if any) is the function

$$f(z) = (z - \bar{z}) \operatorname{Re}(z) + z \operatorname{Im}(z)$$

differentiable

(b) Find $f'(z)$ for these z .

Sol (a) $z = x + iy$

$$\bar{z} = x - iy$$

$$\begin{aligned} z - \bar{z} &= iy - (-iy) \\ &= 2iy \end{aligned}$$

$$\operatorname{Re}(z) = x$$

$$\operatorname{Im}(z) = y$$

$$\begin{aligned} \text{So } f(z) &= (2iy)x + (x+iy)y \\ &= (2xy)i + xy + iy^2 \\ &= [xy] + i[2xy + y^2] \end{aligned}$$

$$u(x, y) = xy$$

$$v(x, y) = 2xy + y^2$$

$$\frac{\partial u}{\partial x} = y$$

$$\frac{\partial v}{\partial y} = 2x + 2y$$

$$\frac{\partial u}{\partial y} = x$$

$$\frac{\partial v}{\partial x} = 2y$$

So $y = 2x + 2y$ (A) $\left(\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \right)$

and $x = -2y$ (B) $\left(\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right)$

must hold.

From (A) $y = -2x$

From (B) $y = -\frac{1}{2}x$

or $-2x = -\frac{1}{2}x$

or $-\frac{3}{2}x = 0 \Rightarrow x = 0$

and $y = 0$

$\therefore f(z)$ is differentiable

at $x=0$ and $y=0$

i.e. $0+0i$ or $z=0$.

(b) $f'(0)$?

Using $f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$

$$\Rightarrow f'(z) = [y] + i [2y]$$

$$\Rightarrow \underline{f'(0+0i)} = 0 + 0i \quad (= 0).$$