

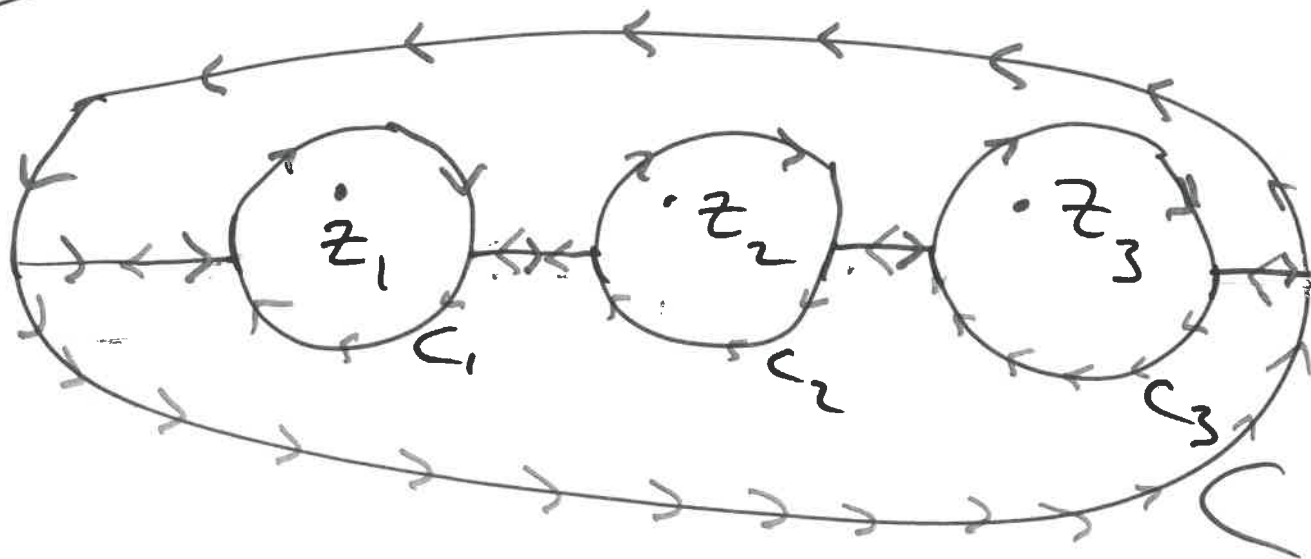
# <sup>lec</sup> 24 Residue theorem

- Let  $f(z)$  be analytic
- inside and on
- a simple closed curve  $C$
- except for a finitely many singular points  $z_1, z_2, \dots, z_k$  inside  $C$

then

$$\oint_C f(z) dz = 2\pi i \sum_{j=1}^k \operatorname{Res}_{z=z_j} f(z)$$

Proof



From Cauchy's Integral Theorem

$$\oint_{\gamma} f(z) dz = 0 \quad \text{AND} \quad \oint_{\gamma} f(z) dz = 0$$

$$\text{or} \quad \oint_{\gamma} f(z) dz + \oint_{\gamma} f(z) dz = 0$$

$$\Rightarrow \oint_{\gamma} f(z) dz - \oint_{\gamma_1} f(z) dz - \oint_{\gamma_2} f(z) dz - \oint_{\gamma_3} f(z) dz = 0$$

$$\Rightarrow \oint_{\gamma} f(z) dz = \oint_{\gamma_1} f(z) dz + \oint_{\gamma_2} f(z) dz + \oint_{\gamma_3} f(z) dz$$

$$\oint_{\gamma} f(z) dz = 2\pi i \operatorname{Res} f(z)_{z=z_1} + 2\pi i \operatorname{Res} f(z)_{z=z_2} + 2\pi i \operatorname{Res} f(z)_{z=z_3}$$

$$\therefore \oint_{\gamma} f(z) dz = 2\pi i \sum_{j=1}^n \operatorname{Res} f(z_j)$$

Ex

$$\oint_{\gamma} \frac{1}{z(z^2+4)(z-i)^2} dz$$

$$\gamma \text{ is } |z-i| = \frac{3}{2}$$

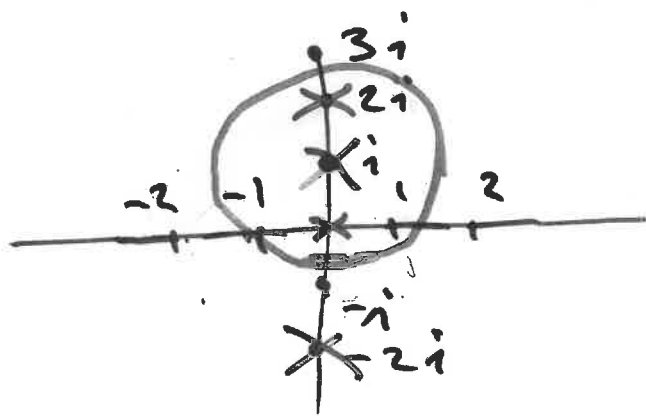
The singularities of the integrand are:

a pole of order 1 at  $z=0$

a pole of order 1 at  $z=2$

a pole of order 1 at  $z=-2$   
 "simple poles"

a pole of order 2 at  $z=i$   
 "Double pole"



$z=0$ ,  $z=i$ ,  $z=2i$  are inside  $C$   
 and  $z=-2i$  is outside  $C$ .

$$\begin{aligned} \text{Res}_{z=0} \frac{1}{z(z^2+4)(z-i)^2} &= \lim_{z \rightarrow 0} \frac{1}{z(z^2+4)(z-i)^2} \cdot z \\ &= \lim_{z \rightarrow 0} \frac{1}{(z^2+4)(z-i)^2} \\ &= \frac{1}{4(-i)^2} = \frac{1}{4 \cdot 2} = \sqrt{-\frac{1}{4}} \end{aligned}$$

$$\text{Res}_{z=2i} \frac{1}{z(z^2+4)(z-i)^2}$$

$$= \text{Res}_{z=2i} \frac{1}{z(z+2i)(z-2i)(z-i)^2}$$

$$= \lim_{z \rightarrow 2i} \frac{1}{z(z+2i)(z-2i)(z-i)^2} \cdot (z-2i)$$

$$= \lim_{z \rightarrow 2i} \frac{1}{z(z+2i)(z-i)^2}$$

$$= \frac{1}{2i(2i+2i)(2i-i)^2}$$

$$= \frac{1}{2i(4i)i^2} = \boxed{\frac{1}{8}}$$

$$\text{Res}_{z=i} \frac{1}{z(z^2+4)(z-i)^2}$$

$$= \lim_{z \rightarrow i} \left[ \frac{1}{z(z^2+4)(z-i)^2} \cdot (z-i)^2 \right]$$

$$= \lim_{z \rightarrow i} \left[ \frac{1}{z(z^2+4)} \right]$$

$$= \lim_{z \rightarrow i} \frac{d}{dz} (z^3 + 4z)^{-1}$$

$$= - (z^3 + 4z)^{-2} (3z^2 + 4) \Big|_{z=i}$$

$$= - (i^3 + 4i)^{-2} (3i^2 + 4)$$

$$= - \frac{-3 + 4}{(-i + 4i)^2}$$

$$= \frac{-1}{-9} = \boxed{\frac{1}{9}}$$

$$\text{Answer} = 2\pi i \left[ -\frac{1}{4} + \frac{1}{8} + \frac{1}{9} \right] = \frac{-\pi}{36} i$$

$$\doteq [0] + i[-.09]$$