

Lec
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Residues and the Residue Theorem

From Cauchy's integral

theorem $\oint_C f(z) dz = 0$

if $f(z)$ is analytic in
a simply connected domain D
which contains C (a simple
closed curve).

If $f(z)$ has a singularity
at $z = z_0$ inside C

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$+ \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots *$$

Recall $\oint_{|z|=1} \frac{dz}{z} = 2\pi i$

Show (using the same method)

$$\oint_{|z-z_0|=R} \frac{dz}{z-z_0} = 2\pi i$$

Show (using the same parameterisation)

$$\oint_{|z-z_0|=R} \frac{dz}{(z-z_0)^k} = 0$$

if $k \in \mathbb{Z} \setminus \{+1\}$.

Integrating ~~\int~~ around z_0

$$\oint_C f(z) dz = \sum_{n=0}^{\infty} a_n \oint_C (z-z_0)^n dz$$

$$+ b_1 \oint_C \frac{dz}{z-z_0} + b_2 \oint_C \frac{dz}{(z-z_0)^2} + \dots$$

$$\Rightarrow \oint_C f(z) dz = b_1 (2\pi i) \quad \text{NB}$$

$f(z)$ has a singularity at z_0 inside \underline{C}

Recall: b_1 is the coefficient of $\frac{1}{z-z_0}$ term.

It is called

the residue of $f(z)$ at $z=z_0$

Ex

$$\oint_{|z|=5} \frac{\sin z}{z^4} dz$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{z} \frac{1}{3!} + \frac{z}{5!} - \dots$$

$$\text{So } b_1 = \frac{-1}{3!} = -\frac{1}{6}$$

As $|z|=5$ contains $z=0$

(the singular point of

$$\left. \frac{\sin z}{z^4} \right)$$

$$\oint_{|z|=5} \frac{\sin z}{z^4} dz = \left(\frac{-1}{6}\right) / (2\pi i) \\ = -\frac{\pi}{3} i$$