

Lec
20

Recall the Taylor series
centred at $z = c$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (z-c)^n$$

Ex

Given the Taylor series
centred around $z=0$ for

$f(z) = e^z$ is:

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

write out the Taylor series
for $f(z) = z^3 e^{z^2}$ with centre $z=0$

One could use the traditional
technique
or noting

$$e^{z^2} = \sum_{n=0}^{\infty} \frac{1}{n!} (z^2)^n$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^{2n}$$

$$\Rightarrow z^3 e^{z^2} = z^3 \sum_{n=0}^{\infty} \frac{1}{n!} z^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} z^{2n+3}$$

Laurent Series

Recall:

The Taylor series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n$$

$$= a_0 + a_1 (z-c) + a_2 (z-c)^2 + \dots$$

Note all the powers are non-negative

The Laurent series is

$$f(z) = \sum_{n=0}^{\infty} a_n (z-c)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-c)^n}$$

$\sum_{n=1}^{\infty} \frac{b_n}{(z-c)^n}$ is known as the principal part
There are formulae to

calculate a_n and b_n ,

however, like in the previous example

there, normally is an easier way

EX (using MacLaurin / Taylor series)

Find the Laurent series

for $f(z) = z^{-5} \sin z$ with
centre 0.

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}$$

$$\Rightarrow z^{-5} \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n-4}$$

$$= z^{-5} \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \frac{z^9}{9!} - \dots \right]$$

$$= \frac{1}{z^4} - \frac{1}{z^2 3!} + \frac{z}{5!} - \frac{z^3}{7!} + \frac{z^4}{9!} - \dots$$

Note: $b_2 = \frac{1}{3!} = \frac{1}{6}$

$$b_4 = 1$$

$$b_n = 0 \quad \text{if } n \neq 2 \text{ AND } n \neq 4.$$

This series holds when $|z| > 0$

We say $f(z)$ has a pole of order 4.

Ex (substitution / Taylor series)

Find the Laurent series

of $z^2 e^{\frac{1}{z}}$ with centre 0.

Sol

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$\Rightarrow e^{\frac{1}{z}} = 1 + \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2!} + \frac{\left(\frac{1}{z}\right)^3}{3!} + \dots$$

$$\Rightarrow z^2 e^{\frac{1}{z}} = z^2 + z + \frac{1}{2!} + \frac{1}{z} \frac{1}{3!} + \frac{1}{z^2} \frac{1}{4!} + \dots$$

$$\left(= \sum_{n=0}^{\infty} \frac{z^2}{z^n n!} = \sum_{n=0}^{\infty} \frac{1}{z^{n-2} n!} \right)$$

This series holds when $|z| > 0$

Recall:

$$\boxed{\text{NB}} \quad \sum_{n=0}^{\infty} w^n = \frac{1}{1-w} \quad |w| < 1$$

Ex

Develop $\frac{1}{1-z}$ in

(a) non-negative powers of z

(b) negative powers of z

Where in the complex plane are the series in (a) and (b) valid?

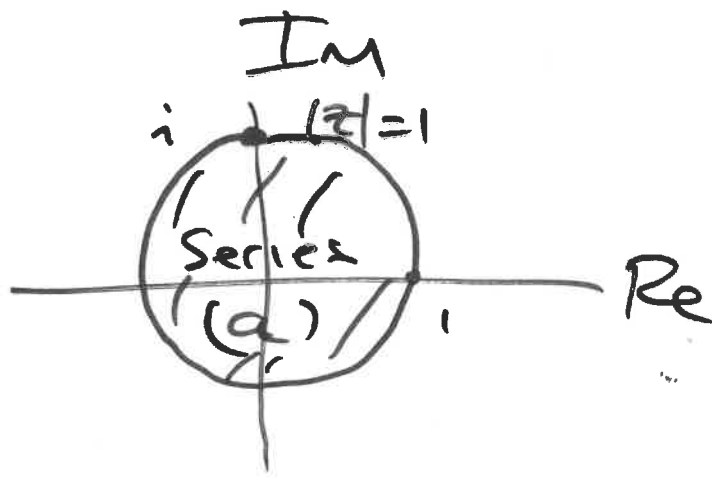
$$(a) \quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{when } |z| < 1$$

This is because

$$\sum_{n=0}^{\infty} z^n = \lim_{n \rightarrow \infty} \frac{1-z^{n+1}}{1-z} = \lim_{n \rightarrow \infty} \frac{1-r e^{i n \theta}}{1-r e^{i \theta}}$$

if $r > 1$, the would equal ∞
So $r < 1$ or $|z| < 1$.

So



Obviously there is no series
for $\frac{1}{1-z}$ when $z=1$

what about when $|z| > 1$?

(b)

$$\frac{1}{1-z}$$

$$= \frac{1}{z} \left(\frac{1}{\frac{1}{z} - 1} \right) \quad \text{NB}$$

$$= -\frac{1}{z} \left(\frac{1}{1 - \frac{1}{z}} \right)$$

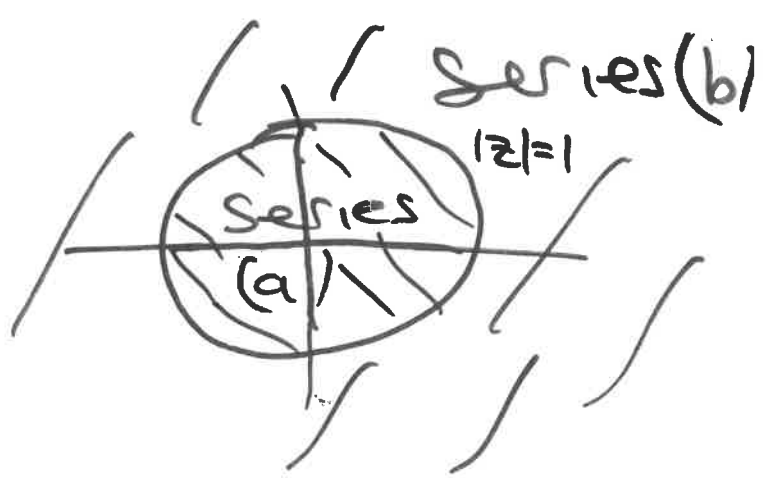
this can be written as

$$-\frac{1}{z} \left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots \right]$$

when $\left|\frac{1}{z}\right| < 1$ or $(|z| > 1)$

$$\text{or } -\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4} - \dots$$

To sum up



Ex

$$f(z) = \frac{1}{z^3 - z^4}$$

$$= \frac{1}{z^3} \left[\frac{1}{1-z} \right]$$

So when $0 < |z| < 1$

$$\frac{1}{z^3} \left[\text{Answer in (a) above} \right]$$

$$= \frac{1}{z^3} \left[\sum_{n=0}^{\infty} z^n \right]$$

$$= \frac{1}{z^3} \left[1 + z + z^2 + \dots \right]$$

$$= \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \dots$$

There is a pole at $z=0$
of order 3.

When $|z| > 1$

$$\frac{1}{z^3} \left[\text{Answer in (b) above} \right]$$

$$= \frac{1}{z^3} \left[-\frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \dots \right]$$

$$= -\frac{1}{z^4} - \frac{1}{z^5} - \frac{1}{z^6} - \dots$$

Ex Expand

$$f(z) = \frac{z}{z-2}$$

in a Laurent series

with centre 0

when (a) $|z| < 2$ (b) $|z| > 2$

(a) Aim to get $f(z)$ in

the form $\frac{1}{1-\ast}$

$$f(z) = z \left[\frac{1}{z-2} \right]$$

$$= -z \left[\frac{1}{2-z} \right]$$

$$= -\frac{z}{2} \left[\frac{1}{1 - z/2} \right]$$

Note this can be expanded
in the usual way if $|z/2| < 1$
i.e. $|z| < 2$

i.e.

$$-\frac{z}{2} \left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \dots \right]$$

$$= -\frac{z}{2} - \frac{z^2}{4} - \frac{z^3}{8} - \dots$$

(b)

$$\frac{z}{z-2} = \frac{1}{z} \left[\frac{z}{1 - 2/z} \right]$$

$$= \frac{1}{1 - \frac{2}{z}}$$

Note this can be expanded
in the usual way if $\left|\frac{2}{z}\right| < 1$
i.e. $|z| > 2$

So $f(z)$

$$= 1 + \frac{z}{z} + \left(\frac{z}{z}\right)^2 + \dots$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{z^n}$$