

Lec
19 Cauchy's integral formula
for derivatives.

- If $f(z)$ is analytic
- in a simply connected domain D
- and C is a simple closed curve in D
- and C encloses the point z_0

then
$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i f^{(n)}(z_0)}{n!}$$

Exercise:

Derive the above by differentiating Cauchy's Integral formula with respect to z_0 .

Ex

$$\oint_C \frac{\cos z}{(z - \pi i)^2} dz$$

Where C is any simple closed curve containing $z_0 = \pi i$

Sol

$$\oint \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{2\pi i f^{(n)}(z_0)}{n!}$$

Let $n = 1$

$$\oint \frac{f(z)}{(z - z_0)^2} dz = \frac{2\pi i f'(z_0)}{1!}$$

$$\text{Let } f(z) = \cos z$$

$$z_0 = \pi i$$

$$\Rightarrow f'(z) = -\sin z$$

$$\begin{aligned} \Rightarrow f'(\pi i) &= -\sin(\pi i) \\ &= -i \sinh(\pi) \end{aligned}$$

$$[\text{As } \sin(iz) = i \sinh(z)]$$

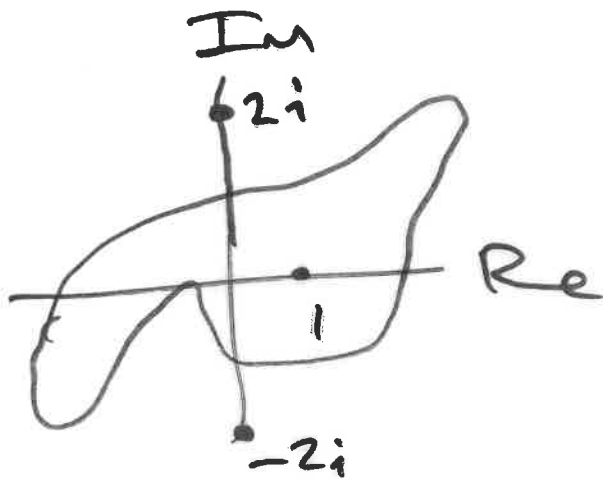
$$\text{So } \oint_C \frac{\cos z}{(z - \pi i)^2} dz = \frac{2\pi i}{1!} [-i \sinh(\pi)]$$

$$= 2\pi \sinh(\pi) \approx 72.6 + 0$$

~~Ex~~ $\oint_C \frac{e^z}{(z-1)^2(z^2+4)} dz$

C is any contour that contains $z=1$ and does not contain $z=2i$ or $z=-2i$.

e.g.



Sol $\oint_C \frac{e^z / (z^2+4)}{(z-1)^2}$

Letting $n=1$, we get

$$\oint_C \frac{f(z)}{(z-z_0)^2} dz = 2\pi i f'(z_0)$$

Let $z_0=1$ and

$$f(z) = \frac{e^z}{z^2+4}$$

$$f'(z) = \frac{(z^2+4)e^z - e^z(2z)}{(z^2+4)^2}$$

$$f'(1) = \frac{5e - e(2)}{25} = \frac{3e}{25}$$

$$\text{Answer} = 2\pi i \left[\frac{3e}{25} \right] = \frac{6\pi e i}{25} \\ (\doteq 0 + 2.05i)$$

EX

$$\oint_C \frac{z^4 - 3z^2 + 6}{(z+i)^3} dz, \quad C \text{ contains } z_0 = -i$$

Letting $n = 2$, we get

$$\oint_C \frac{f(z)}{(z-z_0)^3} dz = \frac{2\pi i f^{(2)}(z_0)}{2!}$$

here $f(z) = z^4 - 3z^2 + 6$, $z_0 = -i$

$$f'(z) = 4z^3 - 6z$$

$$f^{(2)}(z) = 12z^2 - 6$$

$$f^{(2)}(-i) = 12(-i)^2 - 6$$

$$= -12 - 6 = -18$$

$$\text{Answer} = \frac{2\pi i}{2} [-18]$$

$$= -18\pi i$$

$$= [0] + [-56.5]i$$