

Lec 16

The first integration method:

$f(z)$ has to be analytic

Let $f(z)$ be analytic in a

Simply connected domain D ,

then there exists an

indefinite integral of $f(z)$ in D ,

that is, an analytic function $F(z)$

such that $F'(z) = f(z)$ in D

and for all paths in D

joining two points z_0 and z_1

in D

we have

$$\int_{z_0}^{z_1} f(z) dz = F(z_1) - F(z_0)$$

Note:

we can write z_0 and z_1 ,
instead of C

Since we get the same
value for all those
 C from z_0 to z_1 .

Ex 1

$$\begin{aligned} & \int_0^{1+i} z^2 dz \\ &= \left. \frac{z^3}{3} \right|_0^{1+i} \\ &= \frac{1}{3} (1+i)^3 = -\frac{2}{3} + \frac{2}{3}i \end{aligned}$$

Ex 2

$$\int_{-\pi i}^{\pi i} \cos z dz$$

$$= \int_{-\pi i}^{\pi i} \sin z \, dz$$

$$= \sin(\pi i) - \sin(-\pi i)$$

$$= \sin(\pi i) + \sin(\pi i)$$

$$= 2 \sin(\pi i)$$

We had $\sin w = \frac{e^{iw} - e^{-iw}}{2i}$

$$\text{So } 2 \sin(\pi i) = 2 \left[\frac{e^{i(\pi i)} - e^{-i(\pi i)}}{2i} \right]$$

$$= \frac{1}{i} [e^{-\pi} - e^{\pi}] = -i(e^{-\pi} - e^{\pi})$$

$$= i(e^{\pi} - e^{-\pi})$$

$$\doteq [0] + i[23.1]$$

Ex 3

$$\int_{8+\pi i}^{8-3\pi i} e^{z/2} dz$$

$$= 2e^{\frac{z}{2}} \Big|_{8+\pi i}^{8-3\pi i}$$

$$\begin{aligned}
&= 2 \left[e^{\frac{8-3\pi i}{2}} - e^{\frac{8+\pi i}{2}} \right] \\
&= 2 \left[e^{4-\frac{3}{2}\pi i} - e^{4+\frac{\pi}{2}i} \right] \\
&= 2e^4 \left[e^{i(-\frac{3}{2}\pi)} - e^{i(\pi/2)} \right] \\
&= 2e^4 \left[\left\{ \cos\left(-\frac{3}{2}\pi\right) + i \sin\left(-\frac{3}{2}\pi\right) \right\} \right. \\
&\quad \left. - \left\{ \cos\left(\pi/2\right) + i \sin\left(\pi/2\right) \right\} \right] \\
&= 0
\end{aligned}$$

The integrands in the above 3 examples were analytic

over the entire complex plane
 D could equal the complex plane

EX 4

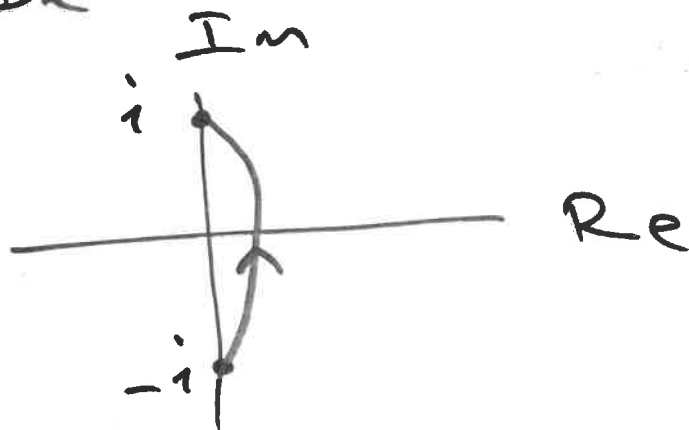
$$\int_{-i}^i \frac{1}{z} dz$$

$\frac{1}{z}$ is not analytic at $z=0$

Here, let D equal

$\left\{ z \in \mathbb{C} \mid \operatorname{Re}(z) \geq 0 \setminus \{0\} \right\}$
This is simply connected.
One integration path

could be



(we are avoiding $z=0$).

$$\int_{-i}^i \frac{dz}{z} =$$

$$\operatorname{Ln} z \Big|_{-i}^i = \operatorname{Ln}(i) - \operatorname{Ln}(-i) *$$

$$\operatorname{Ln} i = \frac{\pi}{2} i$$

Recall!

$$\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) = -i = e^{i\left(-\frac{\pi}{2}\right)}$$

$$\therefore \operatorname{Ln}(-i) = -i\frac{\pi}{2}$$

$$\text{So } * \text{ is } \left(\frac{\pi}{2} i\right) - \left(-i\frac{\pi}{2}\right) = i\pi$$

Note:

$$z = r e^{i\theta}$$

$$\begin{aligned} \text{So } \ln z &= \ln r + \ln e^{i\theta} \\ &= \ln r + i\theta \end{aligned}$$

$$\therefore \ln z = \ln r + i \arg(z)$$

The value of $\ln z$ corresponding to $\text{Arg}(z)$

is denoted $\text{Ln}(z)$

and is called the principal value of $\ln(z)$.

$$\text{So } \text{Ln}(z) = \ln r + i \text{Arg}(z)$$

$$\text{So } \ln(z) = \text{Ln}(z) + 2n\pi i \quad n \in \mathbb{Z}$$

The second integration method:

(This method is not restricted to analytic functions and applies to any continuous complex function.)

- Let C be a piecewise smooth path represented by $z = z(t)$ with $a \leq t \leq b$.

- Let $f(z)$ be a continuous function on C .

Then

$$\int_C f(z) dz = \int_a^b f[z(t)] \dot{z}(t) dt$$

$$\text{Here } \dot{z}(t) = \frac{dz}{dt}$$

Steps for the 2nd method are:

- (A) Represent the path C in the form $z(t)$, $a \leq t \leq b$
- (B) Calculate $\dot{z}(t)$
- (C) Substitute $z(t)$ in for all z in $f(z)$
(i.e. $x(t)$ for x and $y(t)$ for y)
- (D) Integrate $f(z(t))\dot{z}(t)$ from a to b .

EX

$$\oint_{|z|=1} \frac{1}{z} dz \quad \text{An important example.}$$

$|z|=1$ is the unit circle, centre o

Note: We cannot use integration method 1 because to use method 1

" $f(z)$ has to be analytic in a simply connected domain D "

$f(z)$ is not analytic at $z=0$.
(Any region containing $|z|=1$ must contain $z=0$,
If this point is removed e.g.

$$D \text{ is } \frac{1}{2} < |z| < \frac{3}{2}$$

D is not simply connected
(because it has a hole in it.)

Sol

$$(A) \quad z(t) = \cos t + i \sin t$$

$$= e^{it} \quad 0 \leq t \leq 2\pi$$

$$(B) \quad \dot{z}(t) = i e^{it}$$

$$(C) \quad f(z(t)) = \frac{1}{z} = \frac{1}{e^{it}}$$

$$\therefore f(z(t)) = e^{-it}$$

$$(D) \quad \int_0^{2\pi} [e^{-it}] [i e^{it}] dt$$

$$= \int_0^{2\pi} i \, dt$$

$$= i t \Big|_{t=0}^{t=2\pi} = 2\pi i$$

NB

$$\oint_{|z|=1} \frac{dz}{z} = 2\pi i$$

Note if we incorrectly used method 1

the answer = 0.

Ex

Evaluate

$$\int_C (z^2 + 1) \, dz$$

where C is a contour

joining 0 to $2i$ along

(1) the line segment joining 0 to $2i$ parametrized by

$$z(t) = 2it \quad t \in [0, 1]$$

(ii) the semicircle parametrized by

$$z(t) = i + e^{it} \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Sol

$f(z) = \bar{z}^2 + 1$ is not analytic
 \rightarrow use method 2.

$$(i) \quad z(t) = [0] + i[2t]$$

$$\bar{z}^2 + 1 = (x - iy)^2 + 1$$

$$= ([0] - i[2t])^2 + 1$$

$$= -4t^2 + 1$$

$$\therefore f(z(t)) = -4t^2 + 1$$

$$z(t) = 2it$$

$$\dot{z}(t) = 2i$$

$$\int_C f(z) dz = \int_0^1 f(z(t)) \dot{z}(t) dt$$

$$= \int_0^1 [-4t^2 + 1] (2i) dt$$

$$= (2i) \left[-\frac{4t^3}{3} + t \right] \Big|_{t=0}^{t=1}$$

$$= 2i \left(-\frac{4}{3} + 1\right) = -\frac{2}{3}i$$

$$(ii) \quad z(t) = i + e^{it} \quad t \in [-\pi/2, \pi/2]$$

$$z(t) = i + \cos t + i \sin t$$

$$= [\cos t] + i [1 + \sin t]$$

$$\overline{z(t)} = [\cos t] - i [1 + \sin t]$$

$$= \cos t - i \sin t - i$$

$$= e^{-it} - i$$

$$[\overline{z(t)}]^2 + 1 = (e^{-it} - i)^2 + 1$$

$$\dot{z}(t) = ie^{it}$$

$$\int_C (\overline{z}^2 + 1) dz = \int_{-\pi/2}^{\pi/2} [(e^{-it} - i)^2 + 1] ie^{it} dt$$

$$= \int_{-\pi/2}^{\pi/2} [e^{-2it} - 2ie^{-it} - 1 + 1] ie^{it} dt$$

$$= i \int_{-\pi/2}^{\pi/2} [e^{-it} - 2i] dt$$

$$= \int_{-\pi/2}^{\pi/2} (ie^{-it} + 2) dt$$

$$= -e^{-it} + 2t \Big|_{-\pi/2}^{\pi/2}$$

$$\left\{ - \left[\cos(-\pi/2) + i \sin(-\pi/2) \right] + 2(\pi/2) \right\} - \left\{ - \left[\cos(+\pi/2) + i \sin(+\pi/2) \right] - 2(\pi/2) \right\}$$

$$= -\cos \pi/2 + i \sin \pi/2 + \pi$$

$$+ \cos \pi/2 + i \sin \pi/2 + \pi$$

$$= 2\pi + 2i$$

Note different answer to $-z/s$

eventhough

in both cases are integrating from 0 to $2i$.

In the 2nd integration method
 \rightarrow path matters.