

Lec
14 Complex Powers.

Ex

Write i^i in the form $x + iy$, $x, y \in \mathbb{R}$ and plot the results in the complex plane.

Sol

1) Take the \ln and 2) exp of i^i .

$$\begin{aligned} \ln i^i \\ = i \ln i \end{aligned}$$

$$\text{So } i^i = e^{i \ln i} \quad *$$

$$\text{Recall: } e^{i\pi/2} = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} = i$$

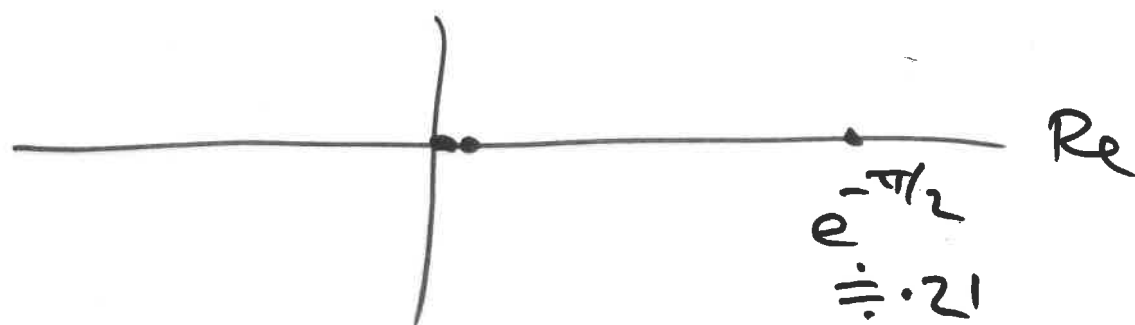
$$\Rightarrow i \frac{\pi}{2} = \ln i$$

$$\text{or } i \left(\frac{\pi}{2} + 2n\pi \right) = \ln i \quad n \in \mathbb{Z}$$

$$\left[\text{because } e^{i(\pi/2 + 2n\pi)} = \cos \left(\frac{\pi}{2} + 2n\pi \right) + i \sin \left(\frac{\pi}{2} + 2n\pi \right) = i \right]$$

Substituting the above into ~~*~~, we get

$$\begin{aligned} 1 &= e^{i \left[i \left(\frac{\pi}{2} + 2n\pi \right) \right]} \\ &= e^{-\pi/2 - 2n\pi} \end{aligned}$$



Ex

Find all values of

$$(3i)^i$$

and express them in the form $x+iy$, $x, y \in \mathbb{R}$.

$$\begin{aligned} \ln \rightarrow \ln (3i)^i \\ = i \ln(3i) \end{aligned}$$

$$\exp \rightarrow e^{i \ln(3i)}$$

$$\text{So } (3i)^i = e^{i \ln(3i)}$$

$$\begin{aligned}
 \text{or } (3i)^i &= e^{i[\ln 3 + \ln i]} \\
 &= e^{i[\ln 3 + i(\frac{\pi}{2} + 2n\pi)]} \quad n \in \mathbb{Z} \\
 &= e^{-\left(\frac{\pi}{2} + 2n\pi\right)} e^{i \ln 3} \quad \text{"exponential polar form"} \\
 &= e^{-\left(\frac{\pi}{2} + 2n\pi\right)} \left[\cos(\ln 3) + i \sin(\ln 3) \right] \\
 &\quad \text{"polar form"}
 \end{aligned}$$

$$\begin{aligned}
 \text{or } &= \left[e^{-\frac{\pi}{2} + 2n\pi} \cos(\ln 3) \right] + i \left[e^{-\frac{\pi}{2} + 2n\pi} \sin(\ln 3) \right] \\
 &\quad \text{"} x + iy \text{" form } \quad x, y \in \mathbb{R}
 \end{aligned}$$

Ex

Find all values of $(i-1)^{2i}$ and express them in polar form.

Sol

$$\text{Ln} \rightarrow \quad \quad \quad 2i \quad \quad \quad 2i$$
$$\text{Ln}(i-1) = 2i \text{Ln}(i-1)$$

$$\text{exp} \rightarrow \quad \quad \quad 2i \quad \quad \quad 2i \text{Ln}(i-1)$$
$$(i-1) = e$$

$$\text{But, } i-1 = -1+i = \sqrt{2} \left[-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i \right]$$
$$= \sqrt{2} \left[\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right]$$
$$= \sqrt{2} \left[\cos \left(\frac{3\pi}{4} + 2n\pi \right) + i \sin \left(\frac{3\pi}{4} + 2n\pi \right) \right], \quad n \in \mathbb{Z}$$
$$= \sqrt{2} e^{i \left(\frac{3\pi}{4} + 2n\pi \right)}$$

$$\text{So } (i-1) = e^{2i \text{Ln} \left\{ \sqrt{2} e^{i \left(\frac{3\pi}{4} + 2n\pi \right)} \right\}}$$

$$= e^{2i \left[\text{Ln} \sqrt{2} + i \left(\frac{3\pi}{4} + 2n\pi \right) \right]}$$
$$= e^{2i \text{Ln}(\sqrt{2}) - 2 \left(\frac{3\pi}{4} + 2n\pi \right)}$$
$$= e^{i \text{Ln} 2 - \left(\frac{3\pi}{2} + 4n\pi \right)}$$

$$= e^{-\left(\frac{3\pi}{2} + 4n\pi \right)} e^{i \text{Ln} 2}$$
$$= e^{-\frac{3\pi}{2} + 4n\pi} \left[\cos(\text{Ln} 2) + i \sin(\text{Ln} 2) \right], \quad n \in \mathbb{Z}$$

Ex

Show $\sin^{-1}(z) = -i \operatorname{Ln}(iz + \sqrt{1-z^2})$.

Sol

First use

(and show as an exercise) that

$$\sin(w) = \frac{e^{iw} - e^{-iw}}{2i}, \quad w \in \mathbb{C}$$

and define $z \in \mathbb{C}$ such that

$$z = \sin(w)$$

or $w = \sin^{-1}(z)$.

Note:

We are being asked to find an expression for w .

We can write

$$\sin(w) = \frac{e^{iw} - e^{-iw}}{2i}$$

writing z for $\sin w$, we get

$$z = \frac{e^{iw} - e^{-iw}}{2i}$$

$$\text{or } 2iz = e^{iw} - e^{-iw}$$

$$\text{or } 2iz e^{iw} = e^{2iw} - 1$$

$$\text{or } 0 = e^{2iw} - 2iz e^{iw} - 1$$

$$\text{or } 0 = (e^{iw})^2 + (-2iz)(e^{iw}) + (-1)$$

$$\text{So } e^{iw} = \frac{2iz + (-4z^2 + 4)^{1/2}}{2}$$

$$e^{iw} = iz + (-z^2 + 1)^{1/2}$$

Take the log of both sides

$$iw = \ln(iz + (-z^2 + 1)^{1/2})$$

$$\text{or } w = -i \ln(iz + (-z^2 + 1)^{1/2})$$

As required.

$$\text{i.e. } \sin^{-1} z = -i \ln(iz + (-z^2 + 1)^{1/2})$$

EX

Find all the values of

$$\sin^{-1}(2i)$$

Sol $\sin^{-1}(z) = -i \operatorname{Ln} \left(iz + \sqrt{1-z^2} \right)$

We need to find all the

square roots of $1 - (2i)^2$

$$= 1 - (-4) = 5$$

$$5 = 5 [\cos 0 + i \sin 0] = 5e^{i \cdot 0}$$

$$= 5 [\cos(2n\pi) + i \sin(2n\pi)] = 5e^{i2n\pi}, n \in \mathbb{Z}$$

Taking the square root

$$\sqrt{5} e^{in\pi}, n \in \mathbb{Z}$$

So, from above

$$\sin^{-1}(2i) = -i \operatorname{Ln} \left(i(2i) + \sqrt{5} e^{in\pi} \right)$$

$$= -i \operatorname{Ln} \left(-2 + \sqrt{5} e^{in\pi} \right)$$

$$e^{in\pi} = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

$$\underline{n \text{ Even}} \Leftrightarrow +\sqrt{5}$$

$$\sin^{-1}(2i) = -i \operatorname{Ln}(-2 + \sqrt{5})$$

$$= -i \operatorname{Ln} \left[(-2 + \sqrt{5}) e^{i2\pi k} \right] \quad k \in \mathbb{Z}$$

writing $-2 + \sqrt{5} > 0$ in "exponential polar form"

$$= -i \left[\operatorname{Ln}(\sqrt{5} - 2) + \operatorname{Ln}(e^{i2\pi k}) \right]$$

$$= -i \left[\operatorname{Ln}(\sqrt{5} - 2) + 2\pi k i \right]$$

$$= +2\pi k - i \operatorname{Ln}(\sqrt{5} - 2) \quad *$$

$$\underline{n \text{ odd}} \Leftrightarrow -\sqrt{5}$$

$$\sin^{-1}(2i) = -i \operatorname{Ln}(-2 - \sqrt{5})$$

$$= -i \operatorname{Ln} \left([2 + \sqrt{5}] (-1) \right)$$

$$= -i \operatorname{Ln} \left([2 + \sqrt{5}] e^{i(2k+1)\pi} \right) \quad k \in \mathbb{Z}$$

writing $-2 - \sqrt{5}$ in "exponential polar form"

$$= -i \left[\operatorname{Ln}(2 + \sqrt{5}) + i(2k+1)\pi \right]$$

$$= + (2k+1)\pi - i \operatorname{Ln}(2+\sqrt{5})$$

Grouping this set of solutions
with *

$$\left\{ 2\pi k - i \operatorname{Ln}(\sqrt{5}-2) \mid k \in \mathbb{Z} \right\} \cup \left\{ (2k+1)\pi - i \operatorname{Ln}(2+\sqrt{5}) \mid k \in \mathbb{Z} \right\}$$

Exercise:

Show and use:

$$\cos w = \frac{1}{2} (e^{iw} + e^{-iw})$$

to show that

$$\cos^{-1} z = -i \operatorname{Ln} \left[z + \sqrt{z^2 - 1} \right]$$