Lec 13 Description of sets of points in the Argand diagram.

Ex A line segment from $-1-i$ to $1+i$.

We parameterize the line segment by introducing a parameter e.g. t.

Pick two arbitrary values of t e.g 0 and 1.

Find the values of a and b such that

$$z(t) = at + b \qquad a, b \in \mathbb{C}$$

$t=0$ $\quad -1-i = a(0) + b$

$\Rightarrow \quad -1-i = b \quad [= -(1+i)]$

$t=1$ $\quad 1+i = a(1) + b$

or $\quad 1+i = a - 1 - i$

or $\quad 2(1+i) = a$

So $z(t) = 2(1+i)t - (1+i)$

When: $t=0$ we get $-1-i$

$t=1$ we get $1+i$

[Exercise:

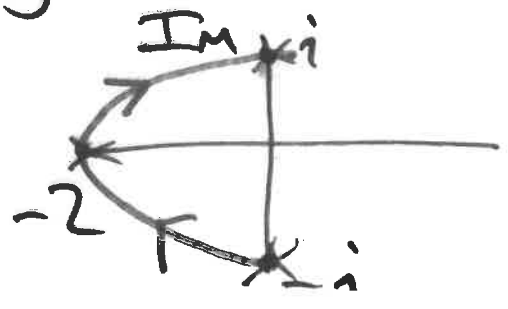
Find the line if you pick $t=-1$ and $t=1$]

The full description of the line segment is:

$$\{ z(t) \in \mathbb{C} \mid z(t) = 2(1+i)t - (1+i), \; t \in [0,1] \}$$

<u>Ex</u>

Describe a parabola in the complex plane going through the points:



i.e. starting at $-i$ and ending at i

$z(t) = at^2 + bt + c \qquad a, b, c \in \mathbb{C}$

Let $t = -1$ when $z = -i$ $\quad$ [$z(-1) = -i$ ...

$z(0) = -2$ and $z(1) = i$.

$z(-1) = -i \Rightarrow -i = a - b + c, \quad -①$

$z(0) = -2 \Rightarrow \boxed{-2 = c} \quad -②$

$z(1) = i \Rightarrow i = a + b + c. \quad -③$

② → ① $\Rightarrow -i = a - b - 2$

② → ③ $\Rightarrow i = a + b - 2$

Add the above 2 equations

$0 = 2a - 4 \Rightarrow \boxed{a = 2}$

As $i = a + b - 2$

$\Rightarrow \boxed{i = b}$

So $z(t) = 2t^2 + it - 2$

The full description is:

$\{ z \in \mathbb{C} \mid z(t) = 2t^2 + it - 2, \; t \in [-1, 1] \}$

If the parameterization has highest power

one : line / line segment

two : parabola

three : cubic polynomial etc

The start and end parameters will tell the start and end points

Ex

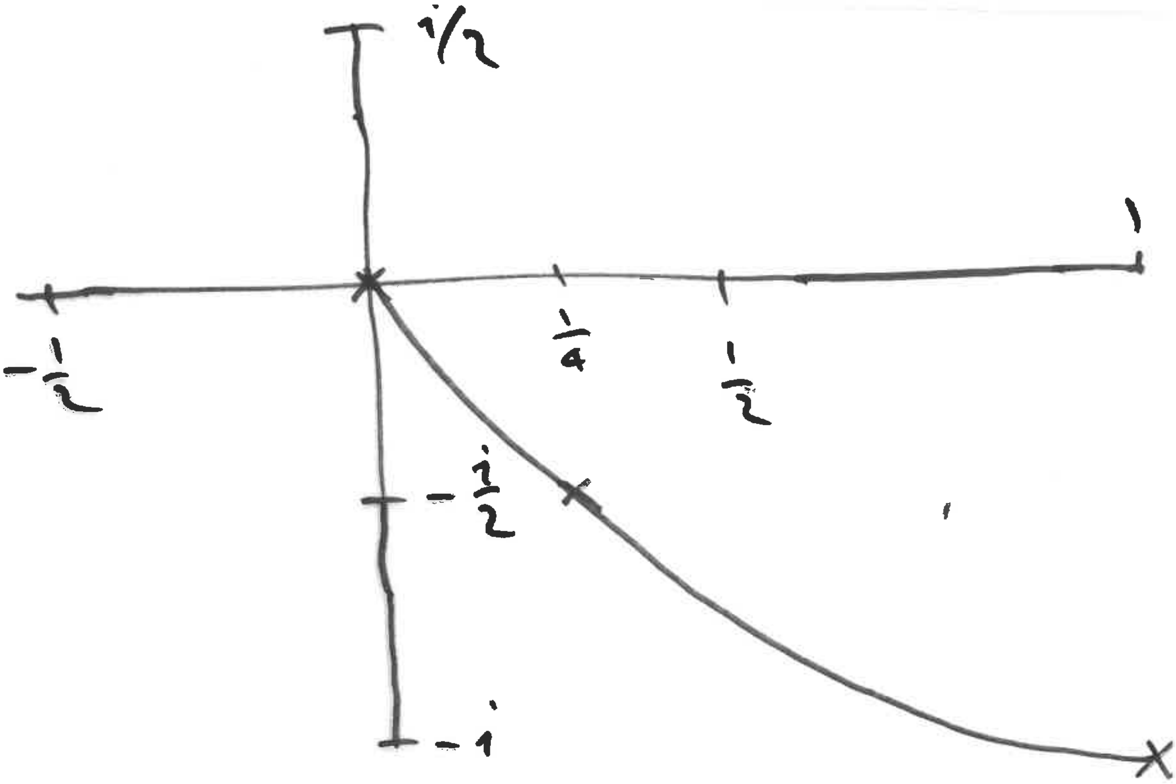
If given

$$\left\{ z(t) = t^2 - it , \quad t \in [0,1] \right\}$$

- it is a parabola,
- it starting point is 0
- it ending point is $1-i$
- it also contains

$$\left(\frac{1}{2}\right)^2 - i\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{i}{2}$$

So



A circle can be described in a number of ways.

<u>First way</u>

<u>Ex</u> $2 = |z - 1 - i|$

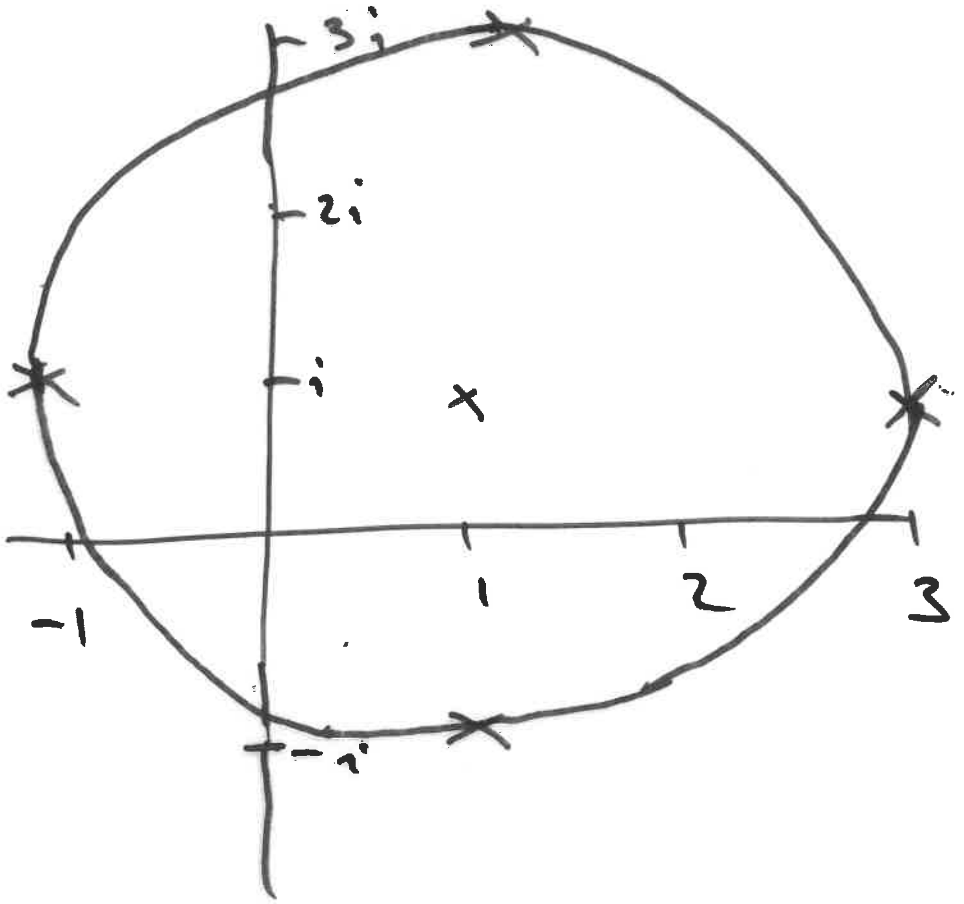
is a circle.

We can write it as

$$2 = |z - (1+i)|$$

The RHS is saying:

" the distance between $1+i$ and z " is 2 units



In general

$$\{z \in \mathbb{C} \mid |z-a| = r\} \quad r \in \mathbb{R}^+$$

is a circle with centre $a \in \mathbb{C}$ and radius r.

$$\{z \in \mathbb{C} \mid |z-a| < r\}$$

is the interior of the circle

$$\{z \in \mathbb{C} \mid |z-a| \leq r\}$$

is the interior of the circle plus the circle.

This is known as a disc

$$\{ z \in \mathbb{C} \mid |z-a| > r \}$$

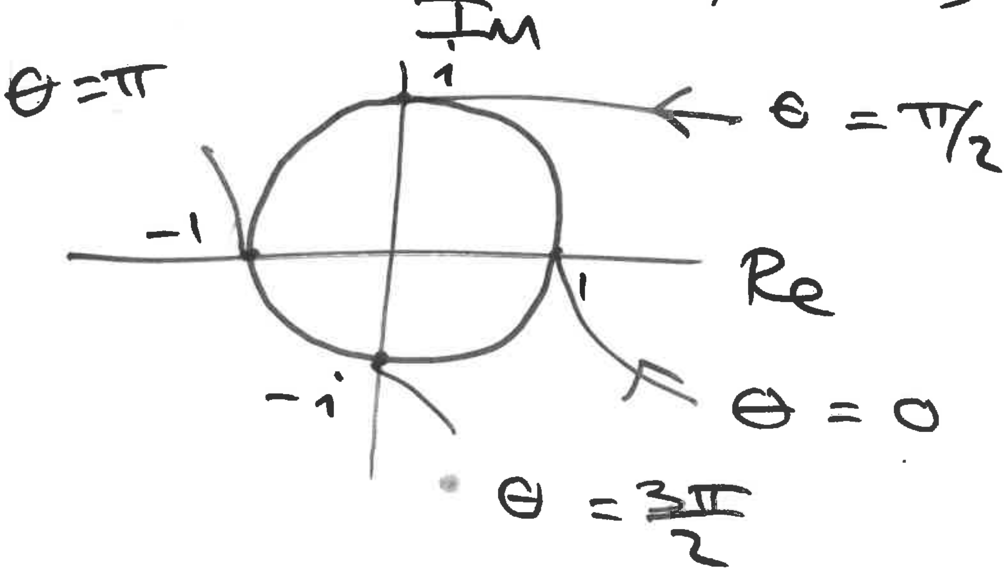
is the exterior of the circle.

<u>Second way</u>

Note:

$$e^{i\theta} = \cos\theta + i \sin\theta.$$

If one let θ run from 0 to 2π and plot the points, we get



we get a circle.

<u>Ex</u>

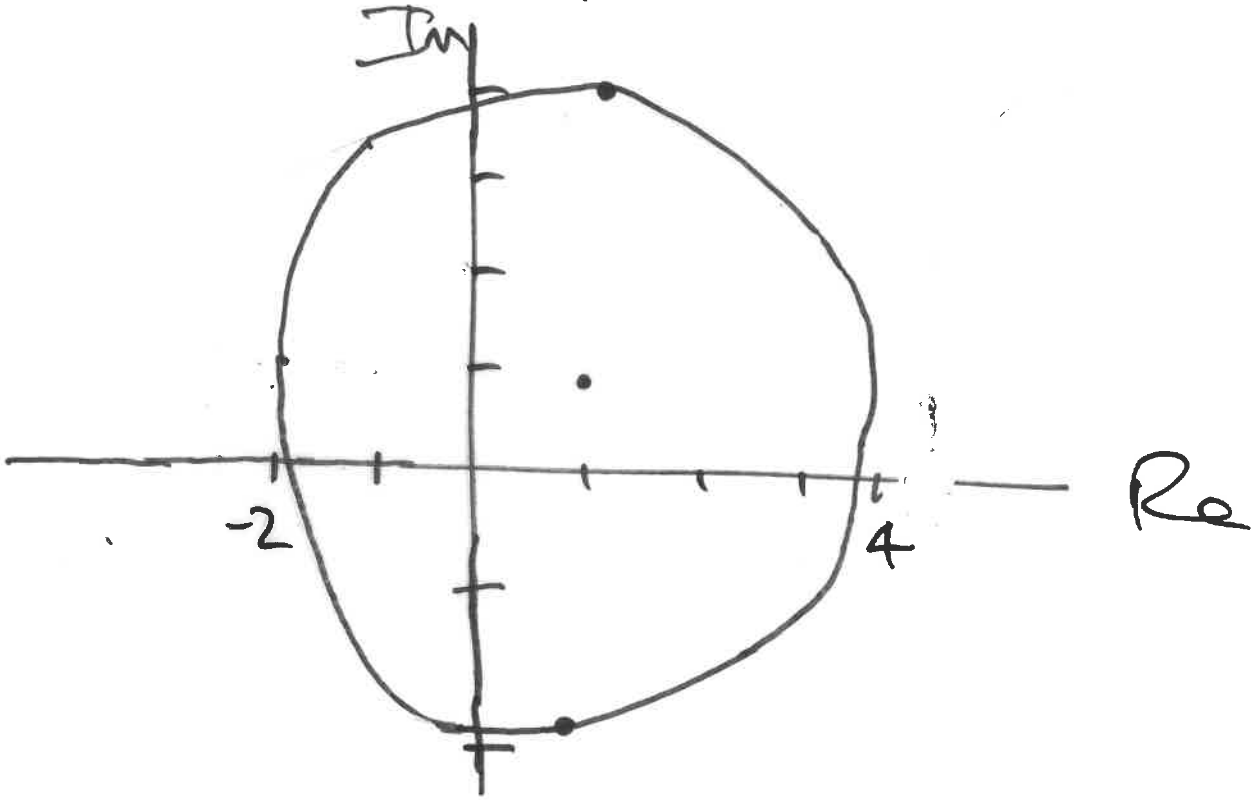
$$2e^{i\theta} \qquad 0 \leq \theta \leq 2\pi$$

is a circle : centre 0 and radius 2.

<u>Ex</u>

$$z(\theta) = 1+i + 3e^{i\theta}$$

is a circle : centre $1+i$ and radius 3.

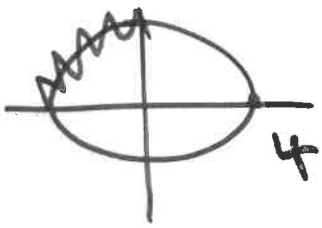


In general a circle of centre $z_0 \in \mathbb{C}$ with radius $r_0 \in \mathbb{R}^+$ can be parameterised by

$$\left\{ z(\theta) = z_0 + r_0 e^{i\theta}, \quad 0 \leq \theta \leq 2\pi \right\}$$

Ex

Parameterise an arc of a circle of centre O and radius 4, that is in the 2^{nd} quadrant.



$$\left\{ z(\theta) = 4e^{i\theta}, \quad \pi/2 \leq \theta \leq \pi \right\}$$

Limits

We treat limits in complex analysis in the same way as we treated them in real analysis.

Ex

Evaluate

$$\lim_{z \to 2i} \frac{z-2i}{z^2+4}$$

we can factorise the denominator

$$\lim_{z \to 2i} \frac{z-2i}{(z-2i)(z+2i)}$$

$$= \lim_{z \to 2i} \frac{1}{z+2i}$$

$$= \frac{1}{2i+2i}$$

$$= \frac{1}{4i} = \frac{i}{4i^2} = -\frac{1}{4} i$$

$$= (0) + \left(-\frac{1}{4}\right) i$$

Ex Evaluate

$$\lim_{z \to -2i} \left(-z^5 - iz\right)$$

$$= -(-2i)^5 - i(-2i)$$

$$= 32 i^5 + 2i^2$$

$$= 32(i^4)(i) + 2(-1)$$

$$= (-2) + (32) i$$