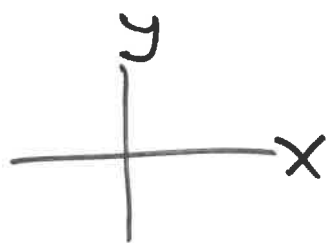


12 Mapping of complex function

In calculus graphs of function $y = f(x)$ in the x - y plane give: — a visual representation — and lead to a better understanding of the function.

This is similar in complex analysis

In real analysis / calculus we needed just 2 dimensions



for a complex function

$$f(z) = u(x, y) + i v(x, y) \quad (z = x + iy)$$

We need 2 dimensions to plot the x and y i.e. complex plane

and we need another
2 dimensions to plot u and v
i.e. a second complex plane.
This complex plane is called
the w plane.

We let $w = f(z) = u + iv$.

We say, f defines a
mapping of points in the
 z plane into the w plane.

We say,

$w_0 = f(z_0)$ is the image
of z_0 with respect to f .

Ex

Mapping $w = z^2$ i.e. $f(z) = z^2$

In polar coordinates, we have

$$z = r (\cos \theta + i \sin \theta) \quad *$$

$$\text{Let } f(z) = R (\cos \phi + i \sin \phi)$$

From *, we have

$$z^2 = r^2 (\cos(2\theta) + i \sin(2\theta))$$

Comparing moduli and arguments

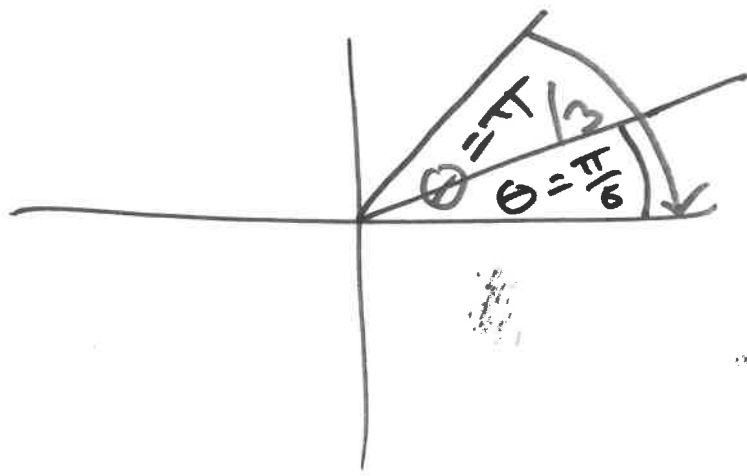
we have

$$R = r^2 \quad \text{and} \quad \phi = 2\theta.$$

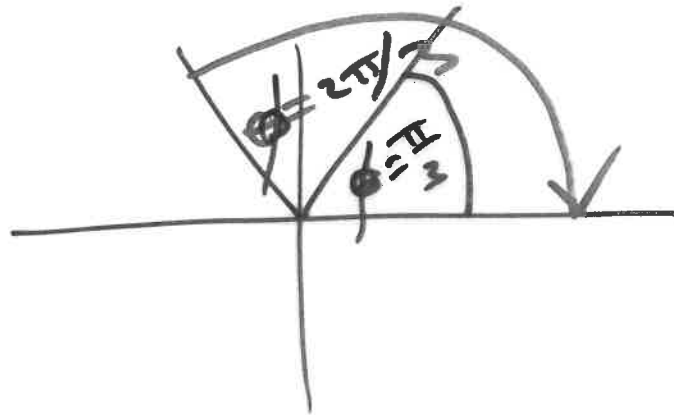
Hence, circles $r = r_0$
are mapped onto circles $R = r_0^2$

Rays $\theta = \theta_0$ are mapped
onto $\phi = 2\theta_0$.

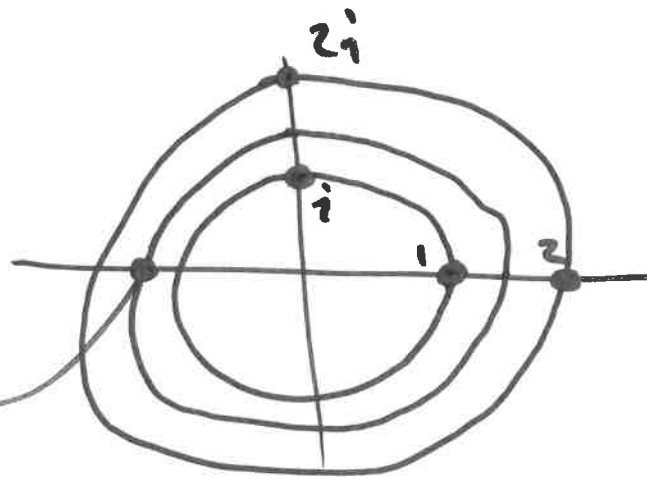
Graphically, we have



z-plane

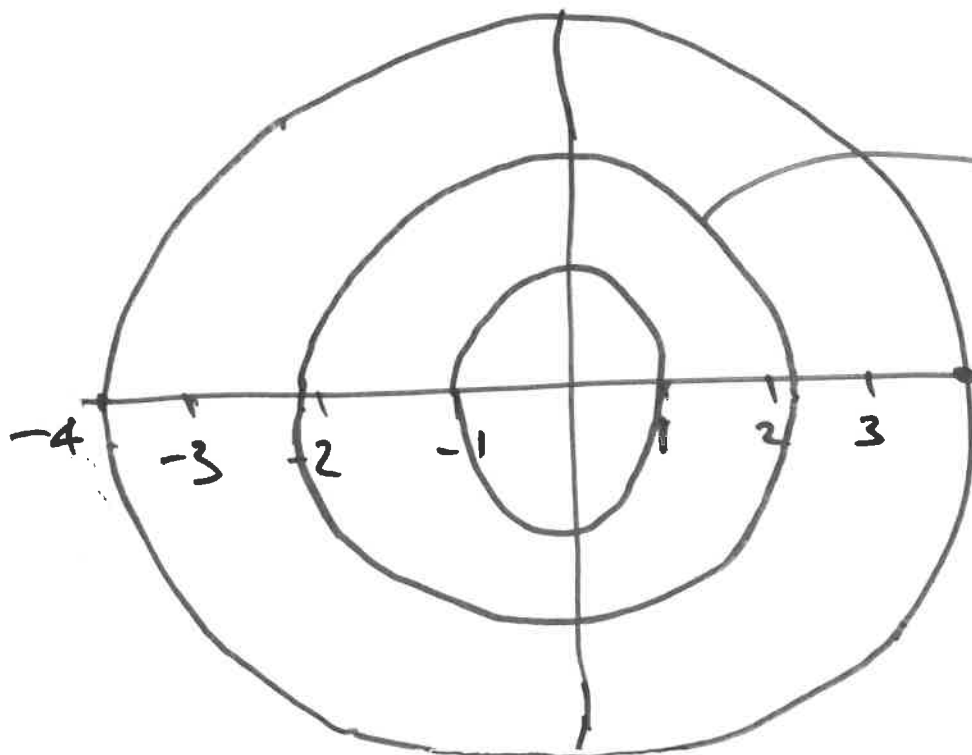


w-plane



z-plane

$r = -1.5$



radius $2.25 = R$

w-plane

The Exponential Function

The complex exponential function is e^z .

Also written as $\exp(z)$.

$$e^z = e^{x+iy} = e^x e^{iy} \quad *$$

$$e^{iy} = \cos y + i \sin y. \quad \underline{\text{NB}}$$

The Maclaurin series for

$$e^* = 1 + * + \frac{*^2}{2!} + \frac{*^3}{3!} + \frac{*^4}{4!} + \frac{*^5}{5!} + \dots$$

$$\text{So } e^{iy} = 1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \dots$$

$$\text{So } e^{iy} = 1 + iy - \frac{y^2}{2!} - \frac{iy^3}{3!} + \frac{y^4}{4!} + \frac{iy^5}{5!} + \dots$$

$$e^{iy} = \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \dots\right) + i \left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \dots\right)$$

$$\text{or } e^{iy} = \cos(y) + i \sin(y) \quad \text{as above}$$

Recall, the polar form of a complex number

$$z = r (\cos \theta + i \sin \theta)$$

This can be written as,

$$z = r e^{i\theta} \quad \underline{\text{NB}}$$

because, we showed (with y instead of θ)

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Important formulae:

$$1) e^{i(\theta + 2n\pi)} = e^{i\theta} \quad n \in \mathbb{Z}$$

because

$$\begin{aligned} e^{i(\theta + 2n\pi)} &= \cos(\theta + 2n\pi) + i \sin(\theta + 2n\pi) \\ &= \cos \theta + i \sin \theta = e^{i\theta} \end{aligned}$$

$$2) e^{i(\frac{\pi}{2})} = i \quad \underline{\text{NB}}$$

because

$$\begin{aligned} e^{i(\frac{\pi}{2})} &= \cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \\ &= (0) + i(1) = i \end{aligned}$$

Exercise: Show

$$e^{\pi i} = -1, \quad e^{-\pi i/2} = -i \quad \text{and} \quad e^{-\pi i} = -1$$

Also

$$\begin{aligned} |e^{iy}| &= |\cos y + i \sin y| \\ &= \sqrt{\cos^2 y + \sin^2 y} = \sqrt{1} = 1 \end{aligned}$$

So

$$|e^z| = |e^{x+iy}| = |e^x| |e^{iy}| = e^x$$

Ex

If $z = 2 + 3\pi i$

find e^z and $|e^z|$

$$\begin{aligned} e^{2+3\pi i} &= e^2 e^{3\pi i} \\ &= e^2 e^{\pi i} = -e^2 \end{aligned}$$

So $e^{2+3\pi i} \approx -7.389 \quad (-e^2)$

$$|e^{2+3\pi i}| = 7.389 \quad (e^2)$$

Ex Write $z_1 = 1+i$ and $z_2 = \sqrt{-i}$
in "exponential polar form".

$$\begin{aligned} z_1 = 1+i &= \sqrt{2} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) \\ &= \sqrt{2} \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right] \\ &= \sqrt{2} e^{i(\pi/4)} \end{aligned}$$

Do z_2 as an exercise.