

Lec
11

Roots of a complex number.

$$\text{As } z = r [\cos \theta + i \sin \theta]$$

We can say

$$z = r [\cos(\theta + 2n\pi) + i \sin(\theta + 2n\pi)]$$

with $n \in \mathbb{Z}$

We can add / subtract any integer multiple of 2π to the argument of the cos and sin functions.

This is because these functions are periodic in 2π .

$$\sqrt[k]{z} = \sqrt[k]{r} \left[\cos\left(\frac{\theta + 2n\pi}{k}\right) + i \sin\left(\frac{\theta + 2n\pi}{k}\right) \right]$$

There are up to k k^{th} roots of z .

To find all the roots let $n = 0, 1, 2, \dots, k-1$.

If, we let

$n = k, k+1, \text{ etc}$

→ the roots are repeated.

Ex

Find the three cube roots of unity (1).

Sol

$$z = 1$$

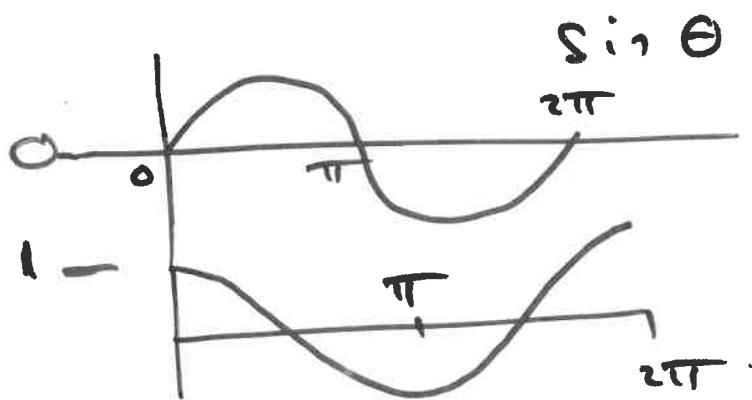
or $z = 1 + 0i$

We need to write z in polar form i.e.

find: r and θ .

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 0^2} = \sqrt{1} = 1$$

$$\text{So } z = 1(1 + 0i)$$



$$\theta = 0, \pi$$

$$\theta = 0, 2\pi$$

For both to hold $\theta = 0$.

$$z = 1 [\cos 0 + i \sin 0]$$

$$z = 1 [\cos 2n\pi + i \sin 2n\pi]$$

$$\sqrt[3]{z} = \sqrt[3]{1} \left[\cos \frac{2n\pi}{3} + i \sin \frac{2n\pi}{3} \right]$$

$$= 1 \left[\cos \frac{2n\pi}{3} + i \sin \frac{2n\pi}{3} \right]$$

$$= \cos \frac{2n\pi}{3} + i \sin \frac{2n\pi}{3}$$

Let $n=0$, we get:

$$\cos 0 + i \sin 0 = \underline{1}$$

Let $n=1$, we get:

$$\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$$

$$= \underline{-\frac{1}{2} + i \frac{\sqrt{3}}{2}}$$

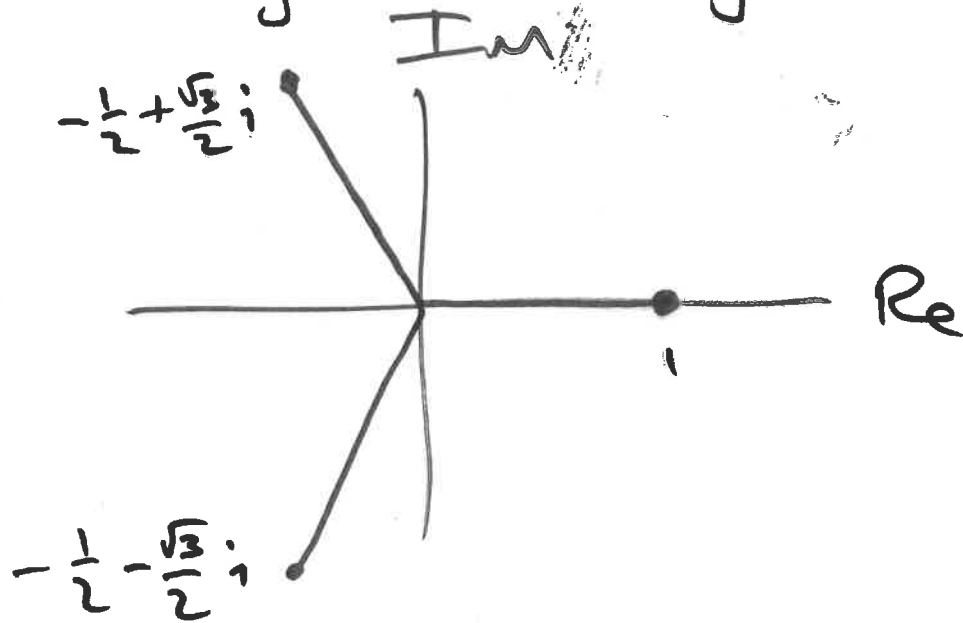
Let $n=2$, we get:

$$\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$$

$$= \underline{-\frac{1}{2} - i \frac{\sqrt{3}}{2}}$$

Ex

Plot the above roots in the Argand diagram.



Exercise:

Check that:

$$z_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

$$z_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}i \quad \text{and} \quad z_3 = 1$$

satisfy $z^3 = 1$.

Ex

Find all the values of

$$(4i - 4)^{1/4}$$

and locate them graphically in the complex plane.

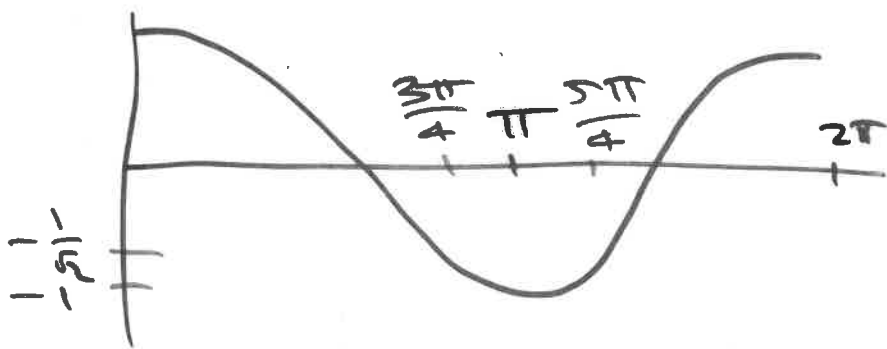
Sol

Write $z = 4i - 4$
in polar form.

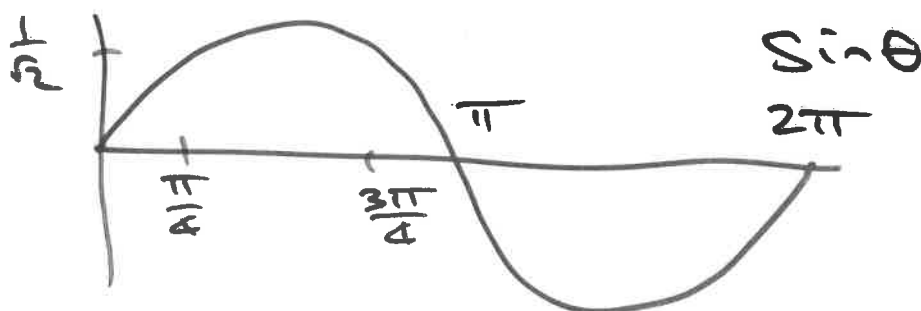
$$\begin{aligned}r &= |z| = \sqrt{4^2 + (-4)^2} \\ &= \sqrt{16 + 16} = \sqrt{32} \\ &= \sqrt{(2)(16)} = 4\sqrt{2}\end{aligned}$$

$$\begin{aligned}\text{So } z &= 4\sqrt{2} \left[\frac{1}{\sqrt{2}}i - \frac{1}{\sqrt{2}} \right] \\ &= 4\sqrt{2} \left[-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \right]\end{aligned}$$

\uparrow $\cos \theta$ \uparrow $\sin \theta$



$$\theta = \frac{3\pi}{4} \text{ or } \frac{5\pi}{4}$$



$$\theta = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}$$

$\theta = \frac{5\pi}{4}$

$$\text{or } z = 4\sqrt{2} \left[\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right]$$

$$\text{or } z = 4\sqrt{2} \left[\cos \left(\frac{3\pi}{4} + 2\pi n \right) + i \sin \left(\frac{3\pi}{4} + 2\pi n \right) \right]$$

$n \in \mathbb{Z}$

$$\Rightarrow z^{\frac{1}{4}} = (4\sqrt{2})^{\frac{1}{4}} \left[\cos \left(\frac{\frac{3\pi}{4} + 2\pi n}{4} \right) + i \sin \left(\frac{\frac{3\pi}{4} + 2\pi n}{4} \right) \right]$$

$$\text{Let } n=0: (4\sqrt{2})^{\frac{1}{4}} \left[\cos \frac{3\pi}{16} + i \sin \frac{3\pi}{16} \right] = z_1$$

$$n=1: (4\sqrt{2})^{\frac{1}{4}} \left[\cos \left(\frac{3\pi}{16} + \frac{\pi}{2} \right) + i \sin \left(\frac{3\pi}{16} + \frac{\pi}{2} \right) \right]$$

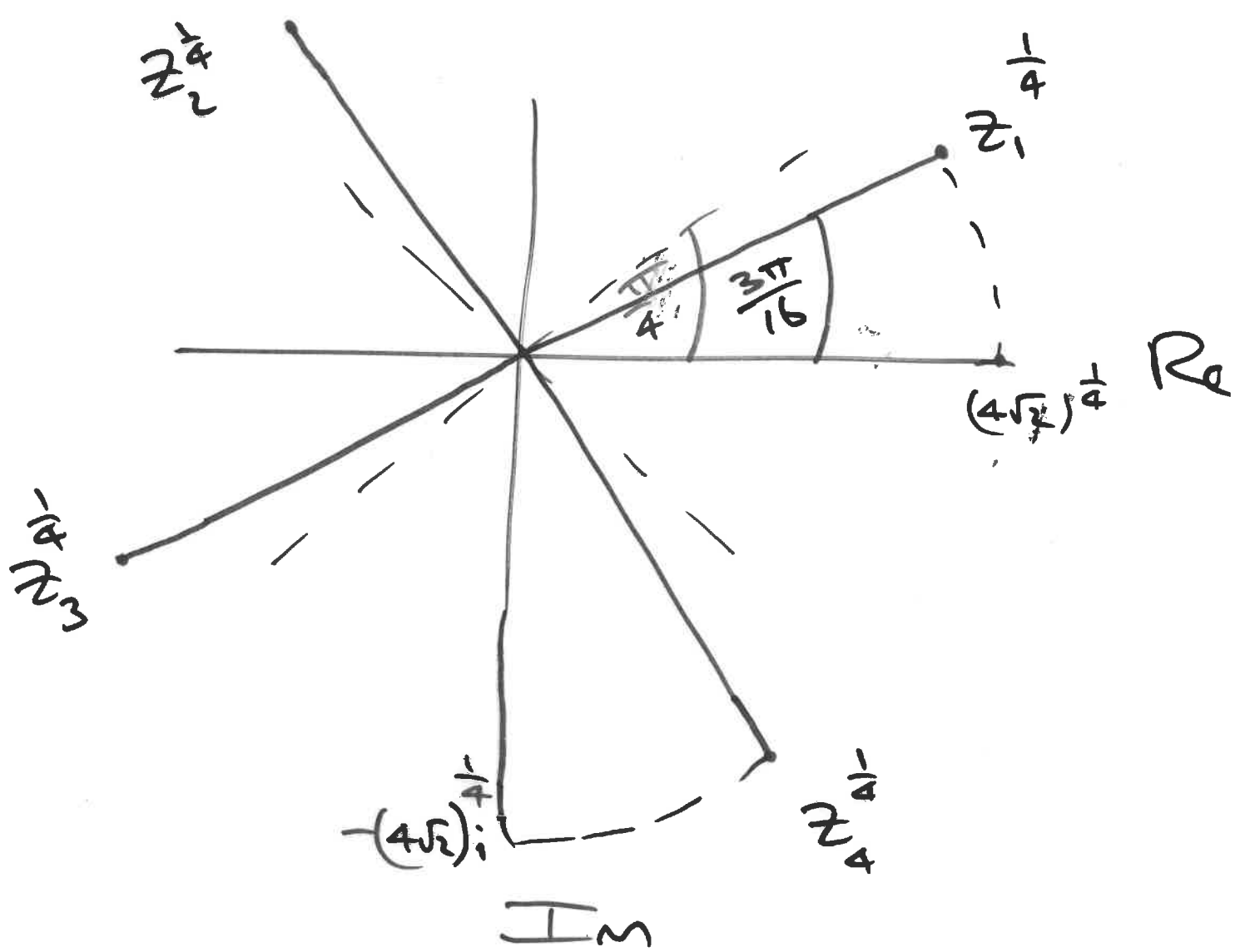
$$\therefore z_2 = (4\sqrt{2})^{\frac{1}{4}} \left[\cos \left(\frac{11\pi}{16} \right) + i \sin \left(\frac{11\pi}{16} \right) \right]$$

$$n=2: (4\sqrt{2})^{\frac{1}{4}} \left[\cos \left(\frac{3\pi}{16} + \pi \right) + i \sin \left(\frac{3\pi}{16} + \pi \right) \right]$$

$$\therefore z_3 = (4\sqrt{2})^{\frac{1}{4}} \left[\cos \left(\frac{19\pi}{16} \right) + i \sin \left(\frac{19\pi}{16} \right) \right]$$

$$n=3: (4\sqrt{2})^{\frac{1}{4}} \left[\cos \left(\frac{3\pi}{16} + \frac{3\pi}{2} \right) + i \sin \left(\frac{3\pi}{16} + \frac{3\pi}{2} \right) \right]$$

$$\therefore z_4 = (4\sqrt{2})^{\frac{1}{4}} \left[\cos \left(\frac{27\pi}{16} \right) + i \sin \left(\frac{27\pi}{16} \right) \right]$$



We can use complex numbers in polar form to solve certain equations.

Ex

Solve for z when
 $z^5 + 243 = 0$
 and plot your answer.

Sol

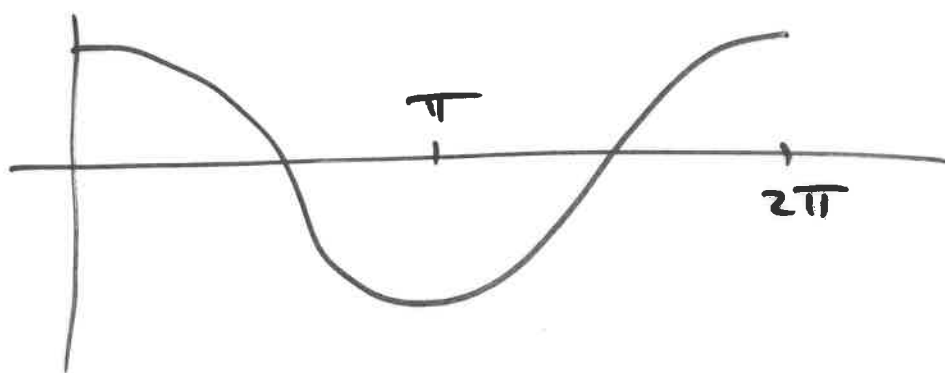
$$z^5 = -243 + 0i$$

i.e. we need to find:

$$z = (-243 + 0i)^{1/5}$$

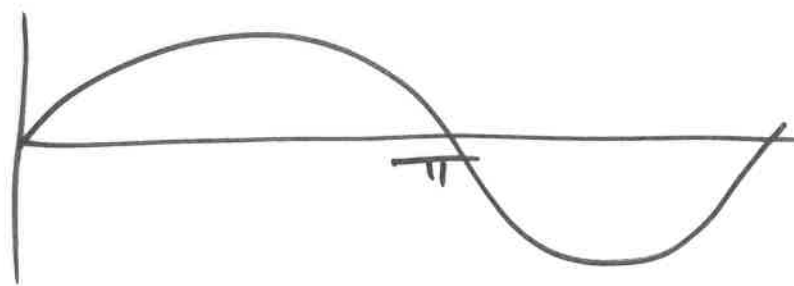
$$|z^5| = \sqrt{(-243)^2 + 0^2} = +243$$

$$\begin{aligned} \text{So } z^5 &= 243 [-1 + 0i] \\ &= 243 [\cos \theta + i \sin \theta] \end{aligned}$$



$\cos \theta$

$$\theta = \pi$$



$$\theta = 0, \pi$$

$$\text{So } \theta = \pi$$

$$z^5 = 243 [\cos \pi + i \sin \pi]$$

$$\text{or } z^5 = 243 [\cos(\pi + 2n\pi) + i \sin(\pi + 2n\pi)]$$

$$n \in \mathbb{Z}$$

$$z = 3 \left[\cos \left(\frac{\pi + 2n\pi}{5} \right) + i \sin \left(\frac{\pi + 2n\pi}{5} \right) \right]$$

$$\text{or } z = 3 \left[\cos \left[\left(\frac{1+2n}{5} \right) \pi \right] + i \sin \left[\left(\frac{1+2n}{5} \right) \pi \right] \right]$$

$$\text{with } n=0: z_1 = 3 \left[\cos \frac{\pi}{5} + i \sin \frac{\pi}{5} \right]$$

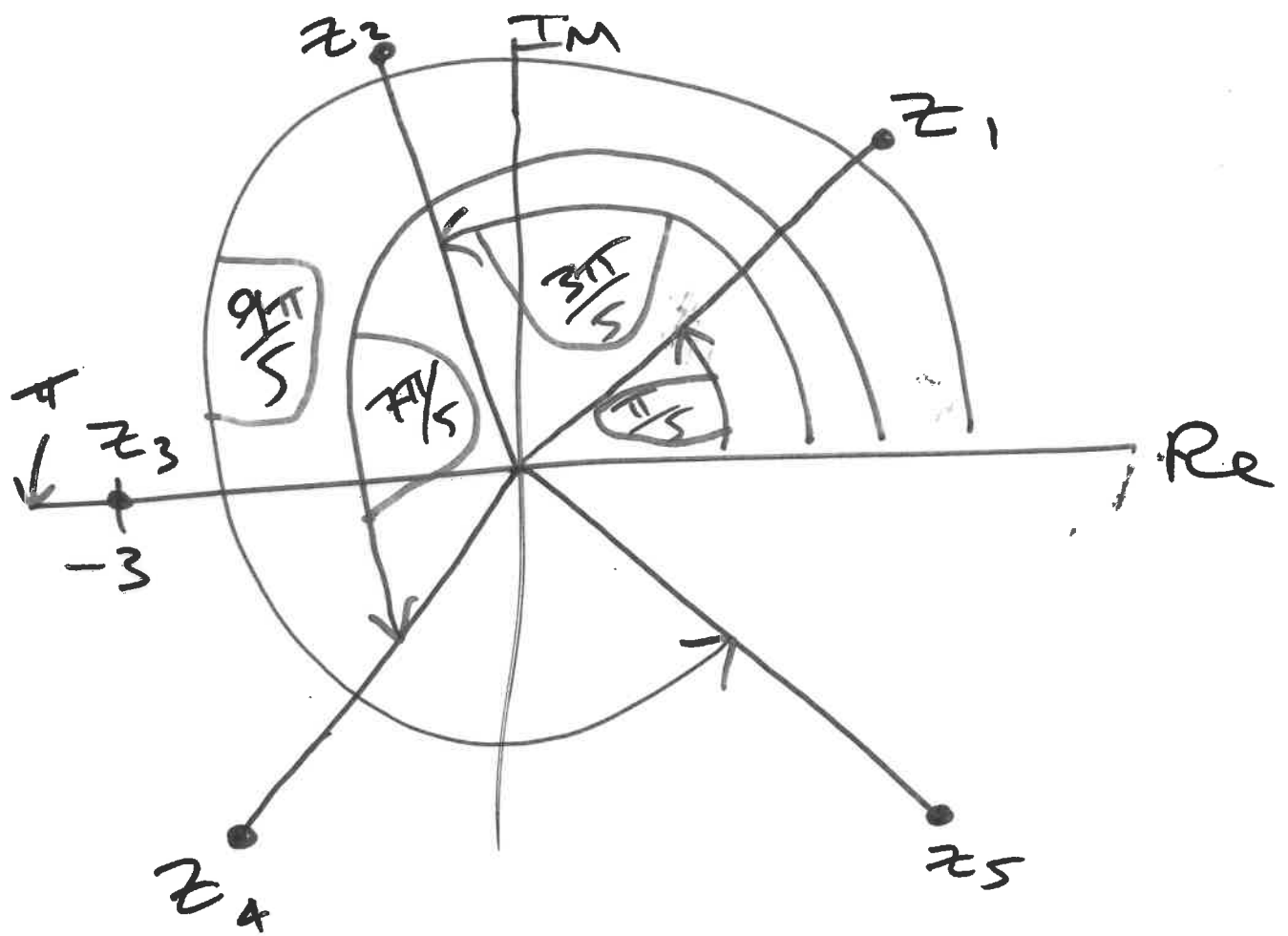
$$n=1: z_2 = 3 \left[\cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5} \right]$$

$$n=2: z_3 = 3 \left[\cos \pi + i \sin \pi \right] = 3(-1) = -3$$

$$n=3: z_4 = 3 \left[\cos \frac{7\pi}{5} + i \sin \frac{7\pi}{5} \right]$$

$$n=4: z_5 = 3 \left[\cos \frac{9\pi}{5} + i \sin \frac{9\pi}{5} \right]$$

Note if you let $n=5$, you get the same result with $n=0$.



Exercise:

(a) convince yourself that all z_1, z_2, z_3, z_4, z_5 satisfy the equation (by finding powers)

(b) convince yourself using geometry (of the complex plane).