

Complex Numbers /

Complex plane.

Many equations have real solutions

e.g. $x^2 = 4 \Rightarrow x = -2$ and $x = 2$.

Other equations do not have real solutions e.g.

$$x^2 = -1.$$

or $x^2 - 10x + 40 = 0$.

We introduce complex numbers to deal with the solutions of the above type of equations

A complex number z is an ordered pair (x, y) of real numbers x and y .

x is called the real part
we write $x = \operatorname{Re}(z)$

y is the imaginary part of

z .

we write $y = \text{Im}(z)$.

Two complex numbers are equal if:

- their real parts are equal
- their imaginary parts are equal

" i " is called the imaginary unit.

$$i = (0, 1)$$

So if $z = (x, y)$

we can write $z = x + iy$
 $x, y \in \mathbb{R}$

i is defined, such that

$$i^2 = -1$$

or $i = \sqrt{-1}$

Addition / Subtraction

$$\text{Let } z_1 = x_1 + i y_1,$$

$$z_2 = x_2 + i y_2$$

$$z_1 + z_2 = (x_1 + x_2) + i (y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + i (y_1 - y_2)$$

Multiplication

$$z_1 \cdot z_2 = (x_1 + i y_1)(x_2 + i y_2)$$

$$= x_1(x_2 + i y_2) + i y_1(x_2 + i y_2)$$

$$= x_1 x_2 + i x_1 y_2 + i y_1 x_2 + \underline{i^2 y_1 y_2}$$

$$\text{As } i^2 = -1$$

$$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i (x_1 y_2 + x_2 y_1)$$

Ex If $z_1 = 8 + 3i$

and $z_2 = 9 - 2i$

$$\text{Re}(z_1) = 8$$

$$\text{Im}(z_2) = -2$$

$$z_1 + z_2 = (8 + 9) + (3 - 2)i$$
$$= 17 + i$$

$$\begin{aligned}
 z_1 \cdot z_2 &= (8 + 3i) \cdot (9 - 2i) \\
 &= [8 \cdot 9 - (3)(-2)] + i[(8)(-2) + 3(9)] \\
 &= [72 + 6] + i[-16 + 27] \\
 &= 78 + i(11)
 \end{aligned}$$

Division.

Again let $z_1 = x_1 + iy_1$; $z_2 = x_2 + iy_2$.

To compute $\frac{z_1}{z_2}$ ($z_2 \neq 0$)

or $\frac{x_1 + iy_1}{x_2 + iy_2}$

or $\frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$

$$= \frac{(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)}{(x_2^2 + y_2^2) + i(0)}$$

$$= \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} \right) + i \left(\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2} \right)$$

Ex if $z_1 = 8 + 3i$
 $z_2 = 9 - 2i$

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{8+3i}{9-2i} \\ &= \frac{8+3i}{9-2i} \cdot \frac{9+2i}{9+2i} \\ &= \frac{(72-6) + i(27+16)}{(81+4)} \\ &= \frac{66}{85} + i \frac{43}{85}\end{aligned}$$

Definition

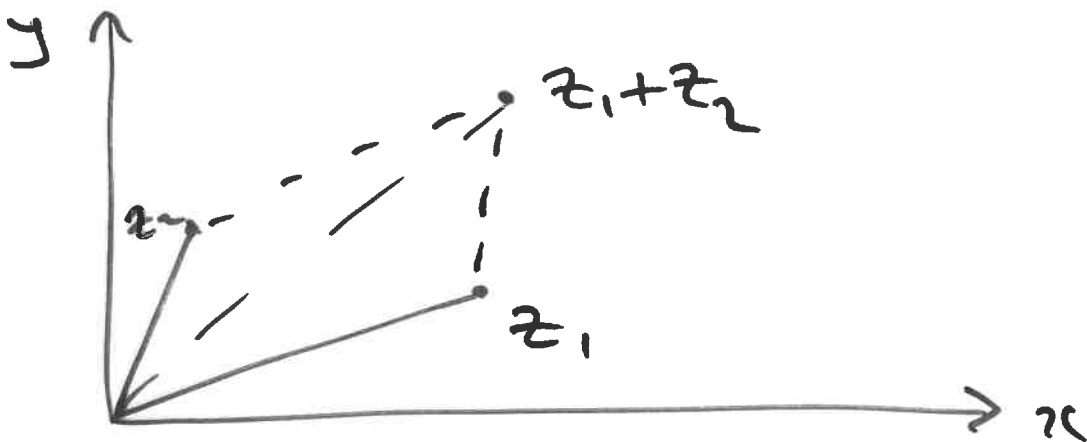
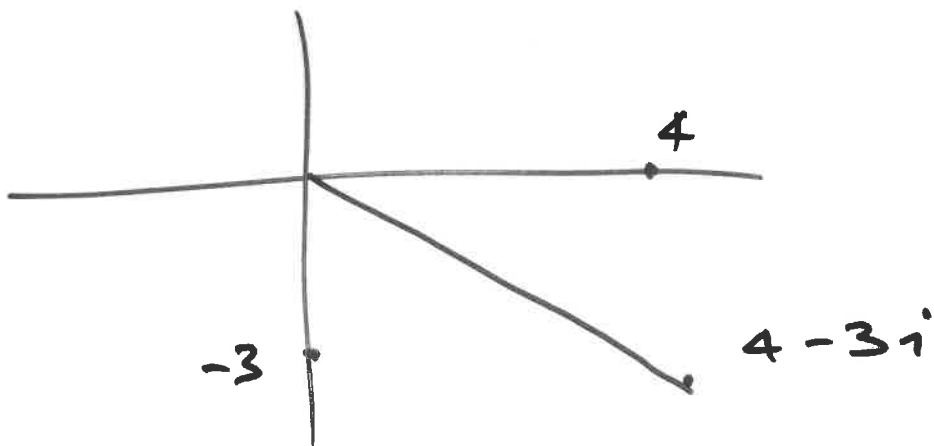
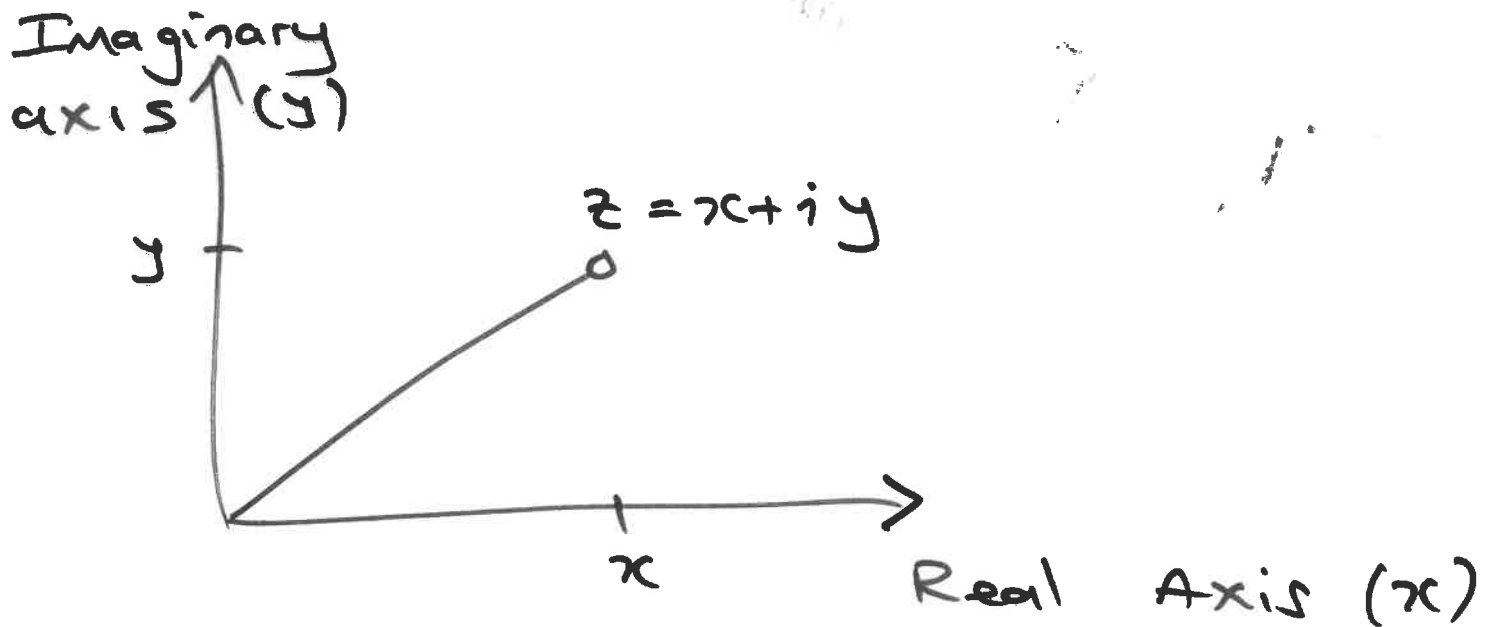
The complex conjugate \overline{z}
of a complex number

$$z = x + iy$$

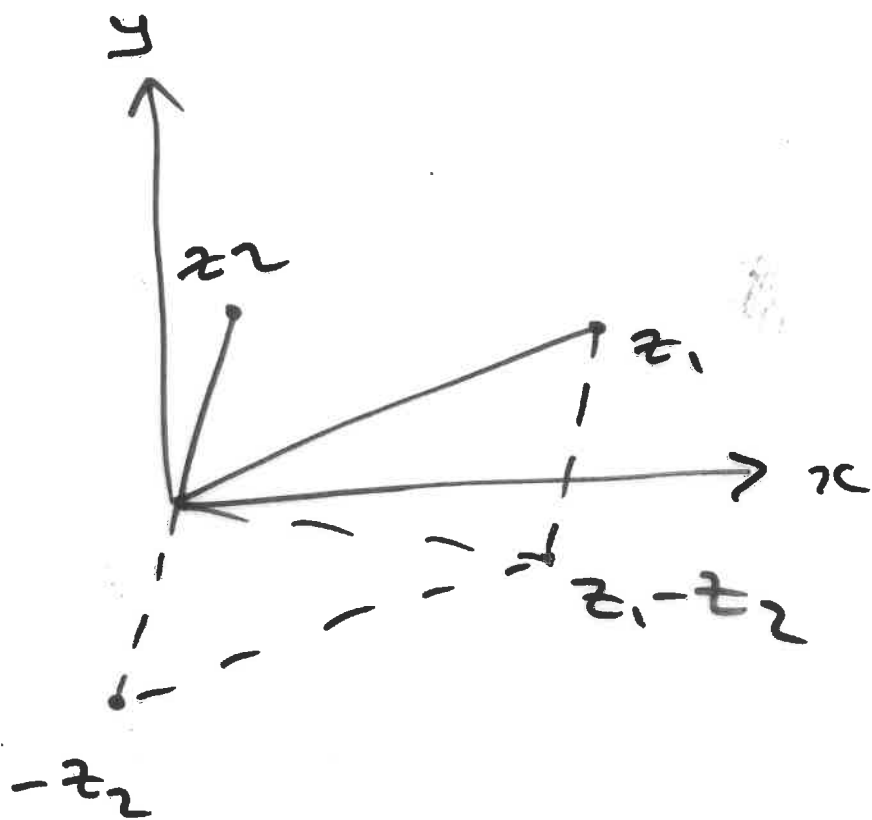
is defined to be by

$$\overline{z} = x - iy.$$

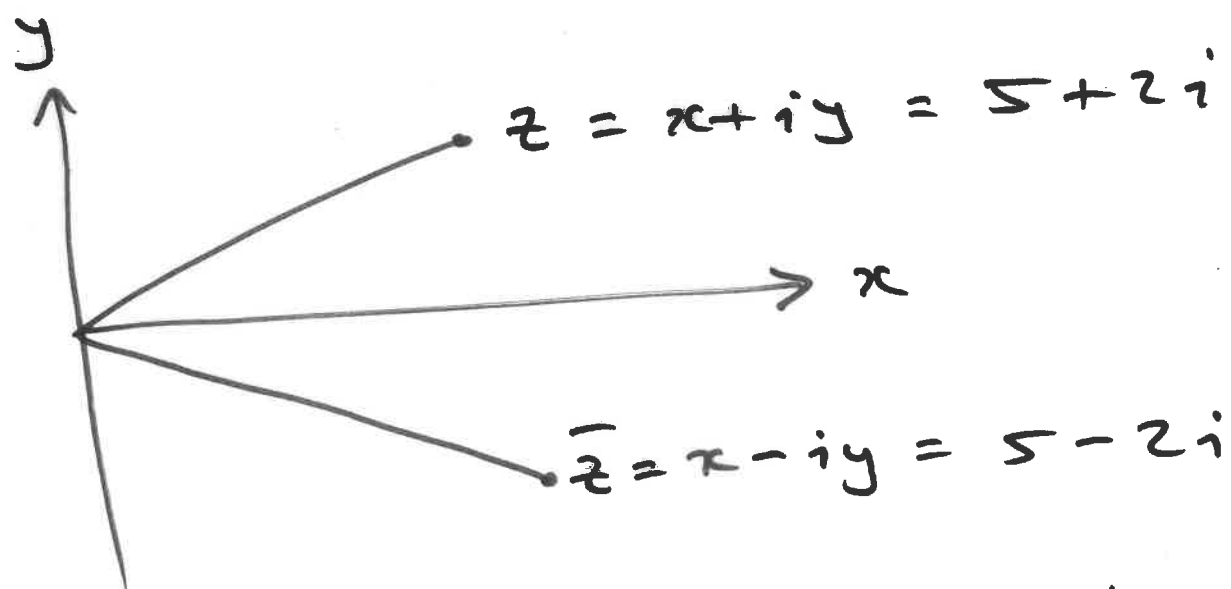
Complex Plane (Argand diagram).



Addition of complex numbers



Subtraction of complex numbers.



i.e. to get the complex conjugate of z in the complex plane \rightarrow reflect z in the real axis

$$z = x + iy \quad \text{--- (1)}$$

$$\bar{z} = x - iy \quad \text{--- (2)}$$

$$\Rightarrow z + \bar{z} = 2x \quad \text{(by adding)}$$

$$\text{or } x = \frac{1}{2} (z + \bar{z}) \quad (= \operatorname{Re}(z))$$

Subtracting (2) from (1), we get

$$z - \bar{z} = 2iy$$

$$\text{or } y = \frac{1}{2i} (z - \bar{z}) \quad (= \operatorname{Im}(z))$$