MA208 Quantitative Techniques for Business

Lecture 21: Linear Programming ctd. - The Simplex Method

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Lecture 21

We learned how to formulate a linear programming problem, and how to solve such a problem in two variables.

Today we will discuss what to do if

- the problem requires an integer solution - and the optimum solution is not an integer, or
- the linear programming problem has more than two variables.

Consider the following problem from Lecture 19.

Example

Example

A distribution firm has to transport 1200 packages using large vans which can take 200 packages each and small vans which can take 80 packages each. The cost of running each large van is €40 and of each small van is €20. Not more than €300 is to be spent on the job. The number of large vans must not exceed the number of small vans.

(i) Formulate this problem as a linear programming problem given that the objective function is to *minimise* costs.

(ii) Solve the linear programming problem to calculate how many large and how many small vans the firm should send.

Example

Solution

(i) $x :=$ number of large vans
$y :=$ " " small "

Minimise $C = 40x + 20y$

subject to

$$200x + 80y \geq 1200$$
$$40x + 20y \leq 300$$
$$x \leq y$$
$$x \geq 0, \quad y \geq 0$$

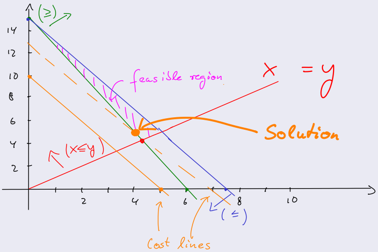
(ii) We sketch the feasible region and see that the optimum solution is not an integer solution!

Example

Solution (ctd.)

We need to identify the integer solutions within the feasible region and select the solution that intercepts the left-most cost line.

Cost $C = 40x + 20y \Rightarrow y = -2x + C/20$



$\Rightarrow$ Solution $x = 4$ and $y = 5$

So Cost $= 40(4) + 20(5) = 260$

The Simplex Method

The corner point method shows that the optimum solution of a linear programming problem is always associated with a corner point of the solution space. The transition from the geometric corner-point solution to the simplex method entails a computational procedure that determines the corner points algebraically.
A main feature of the simplex method is that it solves the linear programming problem in iterations. Each iteration moves the solution to a new corner point that has the potential to improve the value of the objective function. The process ends when no further improvements can be realised.

To explain the method, let's look at an example.

The *Craic & Co.* Company Example

Craic & Co. produce both interior and exterior paints from two raw materials, $M1$ and $M2$. The following table provides the basic data of the problem:

	per ton of Exterior Paint	per ton of Interior Paint	Max. daily availability
Raw Material, $M1$	6 tons	4 tons	24 tons
Raw Material, $M2$	1 tons	2 tons	6 tons
Profit per ton (€1000)	5	4	

They also know that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. Also, the maximum daily demand of interior paint is 2 tons.

Craic & Co. want to determine the optimum (best) product mix of interior and exterior paints that maximises the total daily profit.

Formulating the Model

Let x be the tons produced daily of exterior paints and y be the tons produced daily of interior paints. Then we have the following constraints:

$$\begin{aligned} 6x + 4y &\leq 24 \\ x + 2y &\leq 6 \\ y - x &\leq 1 \\ y &\leq 2 \end{aligned}$$

as well as nonnegativity restrictions $x \geq 0$, $y \geq 0$.

The objective of the company is to maximise

$$p = 5x + 4y.$$

Setting up for the Simplex Method

The right-hand side of the inequality constraints can be thought of as representing the limit on the availability of a resource, in which case the left-hand side would represent the usage of this limited resource by the activities (variables) of the model. The difference between the right-hand side and the left-hand side of the inequality constraints thus yields the unused or **slack** amount of the resource.
The first constraint in our example

$$6x + 4y \leq 24$$

can be converted to

$$6x + 4y + s_1 = 24, \quad s_1 \geq 0.$$

The Simplex Algorithm

Similarly, we can convert the *Craic & Co.* problem as follows.

$$
\begin{aligned}
6x &+4y + s_1 & & & &= 24 \\
x &+2y & +s_2 & & &= 6 \\
-x &+y & & +s_3 & &= 1 \\
 & y & & & +s_4 &= 2
\end{aligned}
$$

The variables s_1, s_2, s_3, s_4 are the slacks associated with the respective constraints.

Next we express the objective function as

$$p - 5x - 4y = 0$$

The Simplex Algorithm

In this manner the starting tableau can be represented as follows:

Basic	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	-5	-4	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

We now need to manipulate the equations in the last tableau so that the Basic-column and the Solution-column will identify the new solution.
This is achieved through Gauss-Jordan row operations (board work).

The Simplex Algorithm

Starting tableau:

	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	-5	-4	0	0	0	0	0
s_1	0	6	4	1	0	0	0	24
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

$R2 \leftarrow \frac{1}{6} R2$

	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	-5	-4	0	0	0	0	0
s_1	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
s_2	0	1	2	0	1	0	0	6
s_3	0	-1	1	0	0	1	0	1
s_4	0	0	1	0	0	0	1	2

$R1 \leftarrow R1 + 5R2$

$R3 \leftarrow R3 - R2$

$R4 \leftarrow R4 + R2$

	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
s_1	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
s_2	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
s_3	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
s_4	0	0	1	0	0	0	1	2

$R3 \leftarrow \frac{3}{4} R3$

The Simplex Algorithm

The new tableau corresponding to the new basic solution (x, s_2, s_3, s_4) thus becomes

Basic	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
x	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
s_2	0	0	$\frac{4}{3}$	$-\frac{1}{6}$	1	0	0	2
s_3	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	0	1	5
s_4	0	0	1	0	0	0	1	2

The Simplex Algorithm

Continuing Gauss-Jordan row operations gives the tableau corresponding to the new basic solution (x, y, s_3, s_4).

	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	0	$-\frac{2}{3}$	$\frac{5}{6}$	0	0	0	20
s_1	0	1	$\frac{2}{3}$	$\frac{1}{6}$	0	0	0	4
s_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	$\frac{5}{3}$	$\frac{1}{6}$	0	1	0	5
s_4	0	0	1	0	0	0	1	2

$R1 \leftarrow R1 + \frac{2}{3} R3$

$R2 \leftarrow R2 - \frac{2}{3} R3$

$R4 \leftarrow R4 - \frac{5}{3} R3$

$R5 \leftarrow R5 - R3$

	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	0	0	$\frac{3}{4}$	$-\frac{1}{2}$	0	0	3
s_1	0	1	0	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_2	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

We get the tableau corresponding to the new basic solution (x, y, s_3, s_4):

Basic	p	x	y	s_1	s_2	s_3	s_4	Solution
p	1	0	0	$\frac{3}{4}$	$\frac{1}{2}$	0	0	21
x	0	1	0	$\frac{1}{4}$	$-\frac{1}{2}$	0	0	3
y	0	0	1	$-\frac{1}{8}$	$\frac{3}{4}$	0	0	$\frac{3}{2}$
s_3	0	0	0	$\frac{3}{8}$	$-\frac{5}{4}$	1	0	$\frac{5}{2}$
s_4	0	0	0	$\frac{1}{8}$	$-\frac{3}{4}$	0	1	$\frac{1}{2}$

Because none of the p-row coefficients associated with the nonbasic variables s_1 and s_2 are negative, this last tableau is optimal.

The optimum solution can be read from the simplex tableau in the following manner.

The optimum values of the variables in the *Basic*-column are given in the right-hand-side *Solution*-column and can be interpreted as

Decision variable	Optimum value	Recommendation
x	3	Produce 3 tons of exterior paint daily
y	1.5	Produce 1.5 tons of interior paint daily
p	21	Daily profit €21,000