MA208 Quantitative Techniques for Business

Lecture 19: Linear Programming

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Section V: Linear Programming

Many problems are concerned with optimising the performance of a system. Here we look at a special class of such problems, namely those in which we optimise a linear objective function subject to linear constraints. These problems are named linear programming problems. The technique is used in a wide range of applications, including agriculture, industry, transportation, economics, health systems, and the military. Linear programming is part of a mathematical area called operations research (OR).

In this section we will consider two-variable models and introduce a graphical approach, the Corner Point Method, to solve them. The algebraic Simplex Method can be used to solve linear programming problems with more than two variables.

Lecture 19 - Outline

Today we will

- Revise graphs of linear equations,
- Discuss graphs of inequalities,
- Learn how to set up a linear programming problem.

Linear equations

Linear programming problems consist of linear equations and inequalities. So let's first revise what we know about these and their graphs.

Linear equations

- The general form of a linear equation is

$$Ax + By + C = 0,$$

  where A, B and $C \in \mathbb{R}$ are constant.
- The graph of a linear function is a line.
- The general form can be rewritten as

$$y = mx + d,$$

  where d is the y-intercept and m is the slope of the line.

Linear equations

Example

Consider the linear equations

(i) $y = x + 1$,

(ii) $y + 2x = 3$.

What are the slope and the y-intercept of each the graphs of the equations? Draw these graphs.

In general we have:

- If the slope of a line is positive (i.e. $m > 0$), then its graph is increasing.
- If the slope of a line is negative (i.e. $m < 0$), then its graph is decreasing.
- If the slope of a line equal to zero (i.e. $m = 0$), then its graph is parallel to the x-axis.

Graphing a linear equation

To graph a linear equation we need at least two points. We can choose any two points, but it is often easier to get the intercepts.

Example

Consider the function $-2x + y = 4$.

1. Calculate the y-intercept.
2. Calculate the x-intercept.
3. Draw the graph of $-2x + y = 4$.

Finding the intersection of two lines

Example

Find the intersection of the lines $y = x + 2$ and $y = -2x + 3$.

1. Use a graphical method to find the point of intersection.
2. Now use an algebraical method to find the point of intersection.

Compare your results!

Exercise

Find the point of intersection of the following two lines:

$$y = 2x + 1$$
$$y = x - 2$$

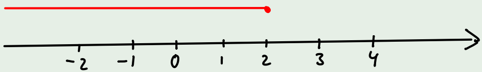
Use a method of your choice.

Graphing Linear Inequalities

Example

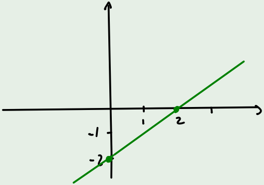
- Consider $x \leq 2$.

  We can represent this interval on a number line.



- How do we graph $y > x - 2$?

  First, graph the line $y = x - 2$.



Intercepts are $(0, -2)$ and $(2, 0)$.

The values $y > x - 2$ lie either above or below the line.

Graphing Linear Inequalities

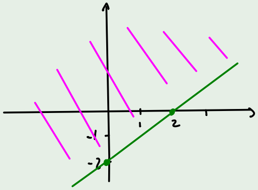
Example

To find out which, try one value, for example $(0, 0)$:

$$0 > 0 - 2.$$

So $(0, 0)$ lies in the required area.

Thus, the graph of $y > x - 2$ is the shaded area below:



Graphing Linear Inequalities

Graphing a linear inequality

To graph an inequality $Ax + By < C$ or $Ax + By > C$,

1. Draw the graph of $Ax + By = C$.
2. Choose a test point on one side of the line and substitute the coordinates into the inequality.
3. Does the test point satisfy the inequality?
   - If so, shade the half-plane that contains the point.
   - If not, shade the other side.

Exercise

Identify the region of the plane given by $2x - 3y \leq 6$.

Formation of Linear Programming problems

Inequalities can be used to illustrate the formation of a linear programming problem.

Example

Suppose a manufacturer of printed circuits has a stock of 200 resistors, 120 transistors and 150 capacitors and is required to produce two types of circuits.

Type A – requires 20 resistors, 10 transistors and 10 capacitors.
Type B – requires 10 resistors, 20 transistors and 30 capacitors.

If the profit on type A circuits is €5 and that on type B circuits is €12, how many of each circuit should be produced in order to maximise the profit?

Formation of Linear Programming problems

The task here is to formulate this problem as a linear programming problem - I am not asking you to solve it just yet. There are three vital stages in the formulation, namely

(a) What are the unknowns? (Decision Variables)
(b) What are the constraints?
(c) What is the profit/cost to be maximised/minimised? (Objective Function)

What are the Unknowns?
The unknowns here are the number of type A and type B circuits produced; so we define
x = number of type A circuits produced,
y = number of type B circuits produced.

Formation of Linear Programming problems

What are the constraints?

- **Resistors.** Since each type A requires 20 resistors, and each type B requires 10 resistors, we have

$$20x + 10y \leq 200,$$

as there is a total of 200 resistors available.

- **Transistors.** Similarly

$$10x + 20y \leq 120$$

- **Capacitors.** Similarly

$$10x + 30y \leq 150$$

Finally you must state the obvious (but nevertheless important) inequalities $x \geq 0, \;\; y \geq 0$.

Formation of Linear Programming problems

What is the profit?

Since each type A gives €5 profit and each type B gives €12 profit, the total profit is P, where

$$P = 5x + 12y.$$

Formation of Linear Programming problems

We can now summarise the problem as:
Maximise

$$P = 5x + 12y,$$

subject to

$$\begin{aligned} 20x + 10y &\leq 200 \\ 10x + 20y &\leq 120 \\ 10x + 30y &\leq 150 \\ x &\geq 0 \\ y &\geq 0. \end{aligned}$$

This is called a **linear** programming problem since the objective function P and the constraints are all linear in x and y.

Example

Example

A factory makes two types of machine, *A* and *B*. The table below details resource requirements for one of each type of machine.

Resource	Machine *A*	Machine *B*	Monthly availability
Energy	100 kWh	200 kWh	20,000 kWh
Steel	60 kg	80 kg	10,000 kg
Labour	2.5 h	2 h	400 h

The company makes a profit of €48 per machine *A* and €38 per machine *B* produced. How many of each type of machine should the company produce to maximize profit?

- Formulate this problem as a linear programming problem.

Example

Solution

Let x = number of machine A,
y = " " " B.

Maximise $P = 48x + 38y$,

subject to

$$100x + 200y \leq 20,000$$
$$60x + 80y \leq 10,000$$
$$2.5x + 2y \leq 400$$
$$x \geq 0$$
$$y \geq 0.$$

Exercise

Formulate the following situations as linear programming problems.

1. A distribution firm has to transport 1200 packages using large vans which can take 200 packages each and small vans which can take 80 packages each. The cost of running each large van is €40 and of each small van is €20. Not more than €300 is to be spent on the job. The number of large vans must not exceed the number of small vans.
2. The annual subscription for a tennis club is €20 for adults and €8 for juniors. The club needs to raise at least €800 in subsciptions to cover its expenses. The total number of members is restricted to 50. The number of junior members is to be between one quarter and one third of the number of adult members.

Outlook

Next week we will learn methods to solve linear programming problems.