# MA208 Quantitative Techniques for Business

## Lecture 18: Using Gaussian Elimination to solve a linear system

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# Recap

In Lecture 15 we introduced *systems of linear equations* and revised methods to solve them efficiently by eliminating variables.

In Lectures 16/17 we showed how systems of linear equations can be represented by *matrices*. Many problems can be *expressed* in terms of matrices, and *solved* by *operations* on matrices. Matrices can be *added* and *multiplied* (provided they have matching sizes). *Elementary row operations* can be used to find the inverse of a **square** matrix, and the inverse matrix can be used to solve *systems of linear equations*, providing they have the same amount of variables and equations.

# Lecture 18

Today we will learn how *elementary row operations* can be used to solve **any** systems of linear equations: we'll introduce an algorithm called ***Gaussian Elimination***.

Let's revise the example from our last lecture to introduce this algorithm.

Consider the following linear system:

(i) $a + b + c = 4$

(ii) $8a + 2b + c = -1$

(iii) $-a + b + c = 2$

This set of equations is represented by the ***augmented matrix***

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 8 & 2 & 1 & -1 \\ -1 & 1 & 1 & 2 \end{pmatrix}$$

By a sequence of elementary row operations this matrix can be changed to

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{7}{6} & \frac{33}{6} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

We will show later how this transformation has been achieved.

Comparing this with the original set of equations it is clear that the third row of the matrix represents the equation

$$c = 3.$$

Substitution of this value of $c$ into the equation represented by Row 2 gives

$$\begin{aligned} b + \frac{7}{6}c &= \frac{33}{6} \\ b &= \frac{33}{6} - \frac{21}{6} = 2. \end{aligned}$$

Finally Row 1 of the simplified matrix represents the equation

$$\begin{aligned} a + b + c &= 4 \\ a + 2 + 3 &= 4 \\ a + 5 &= 4 \\ a &= -1 \end{aligned}$$

Thus the solution to the linear system is $a = -1, b = 2, c = 3.$

The simplified matrix

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{7}{6} & \frac{33}{6} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

is said to be in the ***row echelon form***.

It represents the same linear system as the original matrix, but it enables us to arrive at the solution to the linear system in just a few calculations as seen before.

We will now learn how to reduce the matrix to row-echelon form by using *Gaussian Elimination*.
Solve the linear system

$$\begin{array}{rcrcrcr} a & + & b & + & c & = & 4, \\ 8a & + & 2b & + & c & = & -1, \\ -a & - & b & + & c & = & 2. \end{array}$$

First, represent the linear system by the augmented matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 8 & 2 & 1 & -1 \\ -1 & -1 & 1 & 2 \end{pmatrix}$$

Now reduce the matrix to *row echelon form* by using *Gaussian Elimination*.

$$\begin{pmatrix} 1 & 1 & 1 & 4 \\ 8 & 2 & 1 & -1 \\ -1 & -1 & 1 & 2 \end{pmatrix} \xrightarrow{R_2 \leftarrow R_2 - 8R_1,\ R_3 \leftarrow R_3 + R_1} \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & -6 & -7 & -33 \\ 0 & 0 & 2 & 6 \end{pmatrix}$$

$$\xrightarrow{R_2 \leftarrow -\frac{1}{6} R_2,\ R_3 \leftarrow -\frac{1}{2} R_3} \begin{pmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{7}{6} & \frac{33}{6} \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

From here we get the solution of the linear system:

- by **back-substitution** (as we did in the beginning of the lecture) or
- by finding the ***reduced row echelon form*** (eliminating values above the main diagonal).

Note: In this course we will use back-substitution.

**Example**

A corporation wants to lease a fleet of 12 airplanes with a combined carrying capacity of 220 passengers. The three available types of planes carry 10, 15 and 20 passengers, respectively. How many of each type of plane could be leased?

# Example

## Solution

Let the three types of planes be $x, y, z$ respectively.

Then we have
$$x + y + z = 12$$
and
$$10x + 15y + 20z = 220.$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 10 & 15 & 20 & 220 \end{array}\right) \quad R2 \leftarrow R2 - 10R1$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 5 & 10 & 100 \end{array}\right) \quad R2 \leftarrow \frac{1}{5}R2$$

[Row Echelon Form]

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 0 & 1 & 2 & 20 \end{array}\right) \quad R1 \leftarrow R1 - R2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -8 \\ 0 & 1 & 2 & 20 \end{array}\right)$$

[Reduced Row Echelon Form]

## Solution

$R1 \Rightarrow x - z = -8$

$R2 \Rightarrow y + 2z = 20$

$x, y, z$ need to be non-negative integers.

$$\left.\begin{array}{l} x = z - 8 \geq 0 \\ y = -2z + 20 \geq 0 \end{array}\right\} \Rightarrow \begin{array}{l} z \geq 8 \text{ and} \\ z \leq 10 \end{array}$$

So $8 \leq z \leq 10$

We get three possible solutions:

| $x$ | 0 | 1 | 2 |
|---|---|---|---|
| $y$ | 4 | 2 | 0 |
| $z$ | 8 | 9 | 10 |

# Example

## Example

The University of Tennessee Lady Volunteers defeated the Rutgers University Scarlet Knights 59 to 46. The Lady Volunteers' scoring resulted from a combination of three-point baskets, two-point baskets, and one-point free throws. There were three times as many two-point baskets as three-point baskets. The number of free throws was one less than the number of two-point baskets. (Source: American National Collegiate Athletic Association)

(a) Set up a system of linear equations to find the numbers of three-point baskets, two-point baskets, and one-point free throws scored by the Lady Volunteers.

(b) Solve your system.

## Solution

Let $x$ be the number of three-point baskets,
$y$ " " " " two-point " ,
$z$ " " " " one-point free throws.

Then we have

$$\begin{aligned} 3x + 2y + z &= 59 \\ 3x - y \phantom{{}+z} &= 0 \\ y - z &= 1 \end{aligned}$$

The augmented matrix is

$$\left(\begin{array}{ccc|c} 3 & 2 & 1 & 59 \\ 3 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right)$$

## Solution

Reduce to Row Echelon Form

$$\left(\begin{array}{ccc|c} 1 & \frac{2}{3} & \frac{1}{3} & \frac{59}{3} \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 14 \end{array}\right)$$

Using Back-Substitution we get

$x = 5, \quad y = 15, \quad z = 14$

The Lady Volunteers scored 5 three-point baskets, 15 two-point baskets and 14 one-point free throws.