MA208 Quantitative Techniques for Business

Lecture 16: Systems of linear equations, Matrices ctd.

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Lecture 16 - Outline

Today we will

- Introduce the identity matrix and the inverse of a matrix,
- Learn how to find the inverse of a 2×2 matrix,
- Learn how to find the inverse of an $n \times n$ matrix,
- Use the inverse matrix to solve a linear system of equations.

Matrix Operations

Identity Matrix

- A matrix is called square matrix if it has the same number of rows as columns, i.e. 2×2, 3×3, ..., $n \times n$ matrices.
- A square matrix for which all terms on the main diagonal are equal to 1 and all other elements are zero is called an identity matrix.
- Every square matrix A has a corresponding identity matrix I such that

$$AI = IA = A.$$

Matrix Operations

Examples

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = ?$$

$$\begin{pmatrix} -1.3 & 2 & 35 & \frac{5}{8} \\ 756 & 0 & 17 & -37 \\ -3 & 27\frac{2}{13} & 72 & 674 \\ -232 & 2 & -13 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = ?$$

Matrix Operations

Note: The identity matrix behaves a lot like the number 1 in "ordinary" multiplication of numbers.

The Inverse Matrix

Let A be a square matrix. We call A^{-1} the inverse matrix of A if

$$AA^{-1} = A^{-1}A = I$$

In some cases the inverse matrix can be used to solve linear systems of equations.

Systems of linear equations

Consider the system of linear equations from last lecture:

$$\begin{array}{ccccc} 2x & + & y & = & 8 \\ x & + & 3y & = & 9 \end{array}$$

We can write system this in matrix form as follows:

$$\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}.$$

We may write this as

$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}, \qquad \text{where } A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Systems of linear equations

The square matrix A has an inverse, A^{-1}. Let us multiply each side of the above equation by A^{-1} from the left.

$$A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 8 \\ 9 \end{pmatrix},$$

giving

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 8 \\ 9 \end{pmatrix}.$$

This states that the values of x and y may be found by multiplying the column vector $\begin{pmatrix} 8 \\ 9 \end{pmatrix}$ by the inverse of A.

How do we find A^{-1}?

Finding the inverse of a 2 x 2 matrix

Inverse of a 2 x 2 matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Example

(i) Find the inverse matrix of $A = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$

(ii) Check your result from (i), i.e. is $A \cdot A^{-1} = I$?

Systems of linear equations, ctd.

Now we can solve the system $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$ from above.

$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}$, so we get

$$A^{-1} = \frac{1}{6-1}\begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}.$$

Let's continue our calculations from above:

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 8 \\ 9 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{pmatrix}\begin{pmatrix} 8 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{15}{5} \\ \frac{10}{5} \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

So the solution is $x = 3$ and $y = 2$ as before.

Finding the inverse of an n x n matrix

Finding A^{-1} for any (i.e. $n \times n$) matrix is more complicated. There are several methods to do this. In this course we will introduce the Gauss-Jordan method.

Gauss-Jordan method

The Gauss-Jordan method finds the inverse A^{-1} of a square matrix A:

$$(A|I) \stackrel{\text{row ops}}{\longrightarrow} (I|A^{-1}).$$

Finding the inverse of an n x n matrix

Row operations

We can

- Swap two rows,
- Multiply a row with a number,
- Add a multiple of one row to another.

Example

Find the inverse of $A = \begin{pmatrix} 4 & -1 \\ -6 & 2 \end{pmatrix}$.

Finding the inverse of an n x n matrix

Solution

$$(A|I) = \left(\begin{array}{cc|cc} 4 & -1 & 1 & 0 \\ -6 & 2 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftarrow \frac{1}{4}R_1} \left(\begin{array}{cc|cc} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ -6 & 2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 6R_1} \left(\begin{array}{cc|cc} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & \frac{3}{2} & 1 \end{array}\right) \xrightarrow{R_2 \leftarrow 2R_2} \left(\begin{array}{cc|cc} 1 & -\frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 1 & 3 & 2 \end{array}\right)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{1}{4}R_2} \left(\begin{array}{cc|cc} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 3 & 2 \end{array}\right) = (I|A^{-1})$$

So $A^{-1} = \begin{pmatrix} 1 & \frac{1}{2} \\ 3 & 2 \end{pmatrix}$.