MA208 Quantitative Techniques for Business

Lecture 15: Systems of linear equations, Matrices

Dr Kirsten Pfeiffer

School of Mathematics, Applied Mathematics and Statistics
NUI Galway

Systems of linear equations can be used to solve resource allocation problems in business and economics. Such systems can involve many equations in many variables.

In this section we will review methods for solving two linear equations in two variables. We will then introduce matrices and matrix operations to develop a method that is suitable for solving linear systems of any size.

Lecture 15 - Outline

Today we will

- Revise how to solve systems of two linear equations in two variables,
- Introduce matrices and matrix operations.

Example

If two adult tickets and one child ticket together cost €8, and if one adult ticket and three child tickets together cost €9, what is the price of each ticket?

Let x be the price of an adult ticket and y be the price of a child ticket. Then we have

$$\begin{aligned} 2x &+ y &= 8 \\ x &+ 3y &= 9 \end{aligned}$$

This is a system of two linear equations in two variables.

Systems of two linear equations in two variables

To solve this linear system we need to find those x and y which satisfy both equations.
So far we have two methods to solve this system:

(i) **Graphical method:** Represent both equations as lines and find the point of intersection of these lines.

(ii) **Algebraical method:** Solve the two equations simultaneously.

We use both methods to solve our linear system from above on board and get **one** solution for the system: $(x, y) = (3, 2)$.

So an adult ticket costs €3 and a child tickes costs €2.

Systems of two linear equations in two variables

In general, a linear system can have

(i) **exactly one solution:** the system is consistent and independent,

(ii) **no solution:** the system is inconsistent,

(ii) **infinitely many solutions:** the system is consistent and dependent.

Our aim is to find a method to solve systems with more than two variables and equations. To avoid too much writing, we use matrices to represent the linear system.

Matrices

A matrix is a rectangular array of numbers consisting of horizontal rows and vertical columns. Many problems can be expressed by matrices, and solved by operations on matrices.

How can matrices be used to represent a linear system?

Example

Consider our linear system from above:

$$\begin{array}{rcrcl} 2x & + & y & = & 8 \\ x & + & 3y & = & 9 \end{array}$$

This set of equations can be represented by the matrix

$$\left(\begin{array}{cc|c} 2 & 1 & 8 \\ 1 & 3 & 9 \end{array}\right)$$

Example

Multiplying *Row 2* by (-2) and adding *Row 2* to *Row 1* gives

$$\left(\begin{array}{cc|c} 2 & 1 & 8 \\ 0 & -5 & -10 \end{array}\right)$$

Row 2 translates to $-5y = -10$, so $y = 2$.

Substituting $y = 2$ into *Row 1* gives

$$\begin{aligned} 2x + 1(2) &= 8 \\ 2x + 2 &= 8 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

So we have $(x, y) = (3, 2)$ as above.

Matrices

An $m \times n$ matrix A is a rectangular arrangement of numbers in m *rows* and n *columns*.

Examples

- A 2×2 matrix is of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,
- $\begin{pmatrix} 8 \\ 9 \end{pmatrix}$ is a 2×1 matrix,
- $(7, 8)$ is a 1×2 matrix,
- $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ is a 3×3 matrix.

Just as real numbers and complex numbers have operations which can be performed on them (e.g. addition, multiplication, subtraction), so too do matrices.

We will now define these operations and illustrate them with examples.

Matrix Operations

Matrix Addition

To add to matrices, we are adding corresponding components.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a+w & b+x \\ c+y & d+z \end{pmatrix}$$

Example

$$\begin{pmatrix} 2 & 3 \\ 8 & 5 \end{pmatrix} + \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2+(-1) & 3+2 \\ 8+2 & 5+1 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 10 & 6 \end{pmatrix}$$

Note: Two matrices A and B can be added if they have the same dimensions. Then $A + B = B + A$ (*Commutative Law*).

The Negative of a Matrix

$$-\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

We have

$$\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and therefore

$$-\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Matrix Operations

Scalar Multiplication

$$s\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix}$$

Examples

1. $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} = 2\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

2. Calculate

$$3\begin{pmatrix} 1 & -2 \\ 3 & 4 \end{pmatrix} = ?$$

Matrix Operations

Multiplying a matrix with another matrix is more complicated. Matrix multiplication is based on the principle of "row by column":

$$(a, b, c) \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz.$$

Note: row $\times$ column $=$ a single number

Example

$$\begin{pmatrix} 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 4 \end{pmatrix} = 1 \cdot 3 + 2 \cdot 4 = 11.$$

Matrix Multiplication

The product of two matrices A and B is the matrix C consisting of all the products of the rows of A with the columns of B.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} aw + by & ax + bz \\ cw + dy & cx + dz \end{pmatrix}.$$

Matrix Operations

Examples

(i)

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} -2 & 5 \\ -6 & 11 \end{pmatrix}$$

(ii)

$$\begin{pmatrix} -2 & 1 \\ 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 6 & 8 \end{pmatrix}$$

(iii)

$$\begin{pmatrix} 1 & 2 & 3 \\ -2 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 2 \\ -1 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 5 \\ 3 & -4 \end{pmatrix}$$

Matrix Multiplication

- The product of an $m \times n$ matrix A and a $p \times q$ matrix B exists if and only if $n = p$.
- In general, $BA \neq BA$, see examples (i) and (ii) from above.

Examples

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -1 & 3 \\ 4 & 1 & 5 \\ 2 & 1 & 3 \end{pmatrix}$.

If possible calculate

(i) $A \cdot B$,

(ii) $B \cdot A$.