MA208 Quantitative Techniques for Business

Lecture 12: Mathematics of Finance ctd.

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Lecture 12 - Outline

Today we will talk about

- Continuous Compound Interest
- Future Value of an Annuity

Continuous Compound Interest

In our last lecture we developed the Simple Interest Formula and the Compound Interest Formula. By looking at examples we observed that the more periods we have the higher the future value.

What would happen to the amount if interest were compounded daily, or every minute, or every second?

Let's see what happens if the number of compounding periods per year, m, increases without bound.

Continuous Compound Interest

$$A = P(1+i)^n$$

Substitute $i = \frac{r}{m}, n = mt$

$$A = P(1+\frac{r}{m})^{mt}$$

Multiply the exponent by $\frac{r}{r}(=1)$

$$A = P(1+\frac{r}{m})^{\frac{m}{r}rt}$$

Let $x = \frac{m}{r}$, then $\frac{1}{x} = \frac{r}{m}$

$$A = P(1+\frac{1}{x})^{rt}$$

Use a law of exponents: $a^{xy} = (a^x)^y$

$$A = P[(1+\frac{1}{x})^x]^{rt}$$

Continuous Compound Interest

If $m \to \infty$, then $x \to \infty$ and
if $x \to \infty$, then $(1 + \frac{1}{x})^x \to e \approx 2.7183$.

So we get the following formula for continuous compound interest

Continuous Compound Interest Formula

If a principal P is invested at an annual rate r (expressed as a decimal), then the amount A at the end of t years is given by

$$A = Pe^{rt}$$

Continuous Compound Interest

Example

What amount will an account have after two years if €5000 is invested at an annual rate of 8%

(i) compounded daily?

(ii) compounded continuously?

Solution

(i) Use Compound Interest Formula: $A = P\left(1+\frac{r}{m}\right)^{mt}$

$P = 5{,}000$, $r = 0.08$, $m = 365$, $t = 2$

$A = 5{,}000\left(1+\frac{0.08}{365}\right)^{(365)(2)}$

$= €\,5{,}867.45$

Continuous Compound Interest

Solution

(ii) Use Continuous Compound Interest formula:

$$A = Pe^{rt}$$

$$A = 5{,}000\, e^{(0.08)(2)}$$

$$= €5{,}867.55$$

Future Value of an Annuity

An annuity is any sequence of equal periodic payments. The amount, or future value, of an annuity is the sum of all payments made plus all interest earned. (N.B. payments are made at the end of each time period, known as an ordinary annuity.)

Example

Suppose you decide to deposit €100 every 6 months into an account that pays 6% p/a compounded semiannually. If you make three deposits, one at the end of each interest payment period, over eighteen months, how much money will be in the account after the last deposit is made?

Note

By analyzing this example we will derive a formula for the future value of an annuity.

Future Value of an Annuity

Solution & Analysis

We can use the compound interest formula $A = P(1+i)^n$ to calculate the interest earned by each deposit over the eighteen month period.

- At the end of the first 6 month period €100 is deposited. This sum earns compound interest at the end of the second and third 6 month period (i.e. twice). It follows then that this €100 deposit grows to

$$A = 100(1.03)^2,$$

by the end of the 18 months.

Future Value of an Annuity

Solution & Analysis ctd.

- At the end of the second 6 month period another €100 is deposited. This sum earns interest at the end of the third 6 month period (i.e. once). It follows then that this €100 deposit grows to
$$A = 100(1.03)^1,$$
by the end of the eighteen months.
- At the end of the third six month period a third deposit of €100 is made. This sum does not earn any interest.

The amount in the account at the end of the 18 months is

$$S = 100 + 100(1.03)^1 + 100(1.03)^2 = €309.09$$

We would like to derive a formula to compute this amount.

Future Value of an Annuity

Solution & Analysis ctd.

$$
\begin{aligned}
S &= 100 + 100(1.03)^1 + 100(1.03)^2 & (1) \\
1.03S &= 100(1.03)^1 + 100(1.03)^2 + 100(1.03)^3 & (2)
\end{aligned}
$$

Now we subtract equation (1) from equation (2) to obtain

$$
\begin{aligned}
1.03S - S &= 100(1.03)^3 - 100 \\
0.3S &= 100((1.03)^3 - 1) \\
S &= \frac{100((1.03)^3 - 1)}{0.03} = €309.09
\end{aligned}
$$

This is the formula we are looking for.

Future Value of an Annuity

Future Value of an Ordinary Annuity

$$FV = PMT\frac{(1+i)^n - 1}{i}$$

where

- $FV =$ future value (amount).
- $PMT =$ periodic payment.
- $i =$ rate per period.
- $n =$ number of payments

with the understanding that payments are made at the end of each period.

Future Value of an Annuity

Example

What is the value of an annuity at the end of 20 years if €2,000 is deposited each year into an account earning 8.5% p/a compounded annually?

Solution

We'll calculate this on Monday :)