Another Application of Solving a Simple System of 2 linear equations gives the defn of the cross product of 2 vectors $u \times v$ where $u = (u_1, u_2, u_3)$ & $v = (v_1, v_2, v_3) \in \mathbb{R}^3$

Recall that the cross product of $u$ & $v$ is the vector denoted by $u \times v$ & it is to be perpendicular to both $u$ & $v$

ie $(u \times v) \cdot u = 0 = (u \times v) \cdot v$ so

if $u \times v := (x_1, x_2, x_3)$ then

(i) $\quad u_1 x_1 + u_2 x_2 + u_3 x_3 = 0$ &

(ii) $\quad v_1 x_1 + v_2 x_2 + v_3 x_3 = 0$

$$\longleftrightarrow \quad \left( \begin{array}{ccc|c} u_1 & u_2 & u_3 & 0 \\ v_1 & v_2 & v_3 & 0 \end{array} \right) \xrightarrow[R_2 \to v_2 R_1 - u_2 R_2]{R_1 \to v_1 R_1 - u_1 R_2}$$

$$\left( \begin{array}{ccc} 0 & \overbrace{v_1 u_2 - u_1 v_2}^{=a} & \overbrace{v_1 u_3 - u_1 v_3}^{=b} \\ \underbrace{u_1 v_2 - v_1 u_2}_{=-a} & 0 & \underbrace{v_2 u_3 - u_2 v_3}_{=c} \end{array} \right) \Longleftrightarrow \begin{array}{l} a x_2 = -b x_3 \\ -a x_1 = -c x_3 \\ (x_3 \text{ free}) \end{array}$$

So $x_2 = -\frac{b}{a} x_3$, $x_1 = \frac{c}{a} x_3$ choose $x_3 = -a$ to get the soln $x_1 = -c$, $x_2 = b$, $x_3 = -a$ ie.

$$(x_1, x_2, x_3) = \left( u_2 v_3 - u_3 v_2,\ -(u_1 v_3 - u_3 v_1),\ u_1 v_2 - u_2 v_1 \right)$$

$$= u \times v$$

Mnemonic: Let $e_1 = (1,0,0)$ $e_2 = (0,1,0)$ & $e_3 = (0,0,1)$

The crossproduct $u \times v$ of $u = (u_1, u_2, u_3)$ & $v = (v_1, v_2, v_3) \in \mathbb{R}^3$

can be computed by evaluation the following determinant

$$\begin{vmatrix} e_1 & e_2 & e_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} =$$

$$e_1 (u_2 v_3 - v_2 u_3) - e_2 (u_1 v_3 - v_1 u_3) + e_3 (u_1 v_2 - v_1 u_2)$$

$$= (u_2 v_3 - v_2 u_3, 0, 0) - (0, u_1 v_3 - v_1 u_3, 0) + (0, 0, u_1 v_2 - v_1 u_2)$$

$$= (u_2 v_3 - u_3 v_2, -(u_1 v_3 - v_1 u_3), u_1 v_2 - v_1 u_2)$$

e.g. $u = (2,1,0)$, $v = (3,2,1)$

$$u \times v = \begin{vmatrix} e_1 & e_2 & e_3 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = e_1 (1-0) - e_2 (2-0) + e_3 (4-3)$$

$$= (1, -2, 1)$$

Note $(1,-2,1) \cdot (2,1,0) = 0$

& $(1,-2,1) \cdot (3,2,1) = 3 - 4 + 1 = 0$

## § Matrices & Linear Transformation

Recall last year a Linear Transformation of the Plane $\mathbb{R}^2$ was defined as a Map or fn $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ s.t.

$$: v \longrightarrow L(v)$$

(i) $L(u + v) = L(u) + L(v) \quad \forall\, u, v \in \mathbb{R}^2$

& (ii) $L(rv) = rL(v) \quad \forall\, v \in \mathbb{R}^2$ & $r \in \mathbb{R}$.

Therefore if $v = (x, y) = x(1,0) + y(0,1)$

Then $L(v) = L(x(1,0) + y(0,1)) =$

$$x\,L((1,0)) + y\,L((0,1))$$

So $L$ is determined By $L((1,0)) := (a, c)$ & $L((0,1)) := (b, d)$ where $a, b, c, d \in \mathbb{R}$

& we encoded these 4 numbers in a $2\times 2$ matrix ie an array of 2 rows & 2 columns

$$\begin{pmatrix} \uparrow & \uparrow \\ L((1,0)) & L((0,1)) \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that $L(v) = L((x,y))$ we obtained as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

$$2\times \cancel{2} \quad \cancel{2}\times 1 \quad = \quad 2\times 1$$

ie $L(v) = (ax+by,\ cx+dy)$
& $L$ sends lines to lines.

THIS ALL GOES THROUGH IN A MORE GENERAL SETTING:

Defn: A mapping (OR function)

$$L : \mathbb{R}^n \longrightarrow \mathbb{R}^m \text{ is Linear if}$$
$$: v \longrightarrow L(v)$$

(i) $L(u+v) = L(u) + T(v) \quad \forall u, v \in \mathbb{R}^n$

& (ii) $L(rv) = r L(v) \quad \forall r \in \mathbb{R}, v \in \mathbb{R}^n$

Again using (i) & (ii) we see

That if $v = (x_1, x_2, \ldots, x_n)$
$= x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$

& we encoded these 4 numbers in a $2\times 2$ matrix ie an array of 2 rows & 2 columns

$$\begin{pmatrix} \uparrow & \uparrow \\ L((1,0)) & L((0,1)) \\ \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

so that $L(v) = L((x,y))$ we obtained as:

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THIS ALL GOES THROUGH IN A MORE GENERAL SETTING:

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Again using (i) & (ii) we see

That if $v = (x_1, x_2, \ldots, x_n)$
$= x_1 e_1 + x_2 e_2 + \ldots + x_n e_n$

where $e_1 = (1,0,\cdots,0)$, $e_2 = (0,1,0\cdots,0)$, ..
$e_n = (0,0,\cdots,1)$

Then $L(v) = x_1 L(e_1) + \cdots + x_n L(e_n)$

$$= \sum_{i=1}^{n} x_i L(e_i)$$ & so $L$

is completely determined by the vector $L(e_1), \ldots, L(e_n) \in \mathbb{R}^m$
as each $L(e_i)$ is a vector in $\mathbb{R}^m$
(n of them in all)
ie m, real numbers we will

label them as follows:

$$L(e_1) := (a_{11}, a_{21}, a_{31}, \ldots, a_{m1}) \in \mathbb{R}^m$$

$$L(e_2) := (a_{12}, a_{22}, a_{32}, \ldots, a_{m2}) \in \mathbb{R}^m$$

$$\vdots$$

$$L(e_i) := (a_{1i}, a_{2i}, a_{3i}, \ldots, a_{mi}) \in \mathbb{R}^m$$

$$\vdots$$

$$L(e_n) := (a_{1n}, a_{2n}, a_{3n}, \ldots, a_{mn}) \in \mathbb{R}^n$$

& store this defining information of $L$ as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

an $\underset{\text{rows}}{m} \times \underset{\text{cols}}{n}$ matrix.

where $e_1 = (1,0,\cdots,0)$, $e_2 = (0,1,0\cdots,0)$, $\cdots$

$e_n = (0,0,\cdots,1)$

Then $L(v) = x_1 L(e_1) + \cdots + x_n L(e_n)$

$= \sum_{i=1}^{n} x_i L(e_i)$ & so $L$

is completely determined by the vector $L(e_1), \ldots, L(e_n) \in \mathbb{R}^m$ as each $L(e_i)$ is a vector in $\mathbb{R}^m$ (n of them in all) ie $m$, real numbers we will label them as follows:

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an $\underset{\text{rows}}{m} \times \underset{\text{cols}}{n}$ matrix.

Remark: When we write for instance

$$L(e_1) = (a_{11}, a_{21}, a_{31}, \dots, a_{m1}) \in \mathbb{R}^m$$
$$\parallel$$
$$a_{11} e_1 + a_{21} e_2 + \cdots + a_{m1} e_m$$

where $e_1 = (\underbrace{1, 0, \dots, 0}_{m \text{ entries}}) \in \mathbb{R}^m$

$\vdots$

$e_m = (0, 0, \dots, 1) \in \mathbb{R}^m$

So implicitly the matrix we get involved using the basis or axes

$(\underbrace{1, \dots, 0}_{n \text{ entries}}), \dots (\underbrace{0, 0, \dots, 1}_{n \text{ entries}}) \in \mathbb{R}^n$

for $\mathbb{R}^n$ & the basis or axes

$(\underbrace{1, \dots, 0}_{m \text{ entries}}), \dots, (\underbrace{0, 0, \dots, 1}_{m \text{ entries}}) \in \mathbb{R}^m$

for $\mathbb{R}^m$.

For this reason we sometimes say that

$$\begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$ is the matrix for $L$ w.r.t.

The standard basis (as above) for $\mathbb{R}^n$ & $\mathbb{R}^m$.

Remarks: When we write for instance

$$L(e_1) = (a_{11}, a_{21}, a_{31}, \dots, a_{m1}) \in \mathbb{R}^m$$

$$\shortparallel$$

$$a_{11} e_1 + a_{21} e_2 + \cdots + a_{m1} e_m$$

Where $e_1 = (\underbrace{1, 0, \dots, 0}_{m \text{ entries}}) \in \mathbb{R}^m$

$\vdots$

$e_m = (0, 0, \dots, 1) \in \mathbb{R}^m$

So implicitly the Matrix we get involved using the Basis or axes

$(\underbrace{1, \dots, 0}_{n \text{ entries}}), \dots, (\underbrace{0, 0, \dots, 1}_{n \text{ entries}}) \in \mathbb{R}^n$

For $\mathbb{R}^n$ & the Basis or axes

$(\underbrace{1, \dots, 0}_{m \text{ entries}}), \dots, (\underbrace{0, 0, \dots, 1}_{m \text{ entries}}) \in \mathbb{R}^m$

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The Standard Basis (as above) for $\mathbb{R}^n$ & $\mathbb{R}^m$.