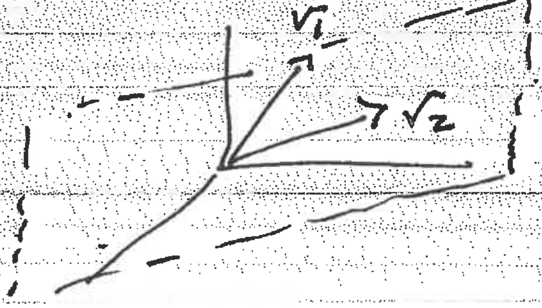
Defn: If $v_1, \dots, v_p$ are vectors in $\mathbb{R}^n$, The set of all linear combinations of $v_1, \dots, v_p$ ie $\{ v = c_1 v_1 + \dots + c_p v_p \mid c_1, \dots, c_p \in \mathbb{R} \}$ is called the subeet of $\mathbb{R}^n$ spanned (or generated) By $v_1, \dots, v_n$ & denoted By $\text{Span}\{v_1, \dots, v_p\}$ or $\langle v_1, \dots, v_p \rangle$

ie a vector $w \in \mathbb{R}^n$ is in the span of $v_1, \dots, v_p$ if the exist a soln of The vector eqn $x_1 v_1 + x_2 v_2 + \dots + x_p v_p = w$

Geometrically: The span of one vector $v = \{ tv \mid t \in \mathbb{R} \}$ ie is The line Through the origin in the direction v.

The span of 2 vectors v_1 & v_2 (not multiples of each other) is the Plane through the origin that they lie in



Ex Prove that the vectors $v_1 = (1,-1,0)$ $v_2 = (1,0,2)$ & $v_3 = (0,1,1)$ are linearly independent. ie Show that the only soln of the vector eqn

$$x_1(1,0,2) + x_2(0,1,1) + x_3(1,-1,0) = (0,0,0)$$

is the trivial soln $x_1 = x_2 = x_3 = 0$

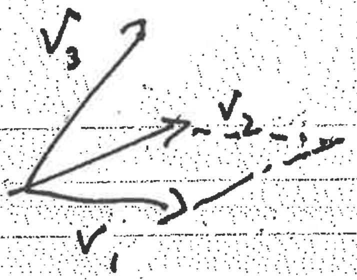
ie
$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 - x_3 &= 0 \\ 2x_1 + x_3 &= 0 \end{aligned} \Longleftrightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 2 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_3 - 2R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \xrightarrow[\substack{(-1)R_3 \\ R_2 - R_3}]{R_1 + R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \Rightarrow x_3 = x_2 = x_1 = 0$$

Geometrically: The vectors v_1, v_2, v_3 don't lie in a plane but point into 3 different "dimensions"

PICTURE: 

So we can use these 3 vectors $v_1 = (1,-1,0)$, $v_2 = (1,0,2)$ & $v_3 = (0,1,1)$ as axes in $\mathbb{R}^3$ instead of $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ & $e_3 = (0,0,1)$ and express any vector as a linear combination of v_1, v_2, v_3.

e.g. $v = (5,-1,7) =$

$$2(1,-1,0) + 3(1,0,2) + 1(0,1,1)$$

By solving the system

$$(5,-1,7) = x_1(1,-1,0) + x_2(1,0,2) + x_3(0,1,1)$$

ie
$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ -1 & 0 & 1 & -1 \\ 0 & 2 & 1 & 7 \end{array}\right) \xrightarrow{R_2+R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 5 \\ 0 & 1 & 1 & 4 \\ 0 & 2 & 1 & 7 \end{array}\right) \xrightarrow[R_3-2R_2]{R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & -1 & -1 \end{array}\right)$$

$$\xrightarrow[R_2+R_3]{R_1-R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & -1 \end{array}\right) \Rightarrow x_3 = 1,\ x_2 = 3$$

& $x_1 = 2$

Another way to say this is that every vector $w \in \mathbb{R}^3$ is in the Span of $v_1 = (1,-1,0)$ & $v_2 = (1,0,2)$ & $v_3 = (0,1,1)$ ie $\mathbb{R}^3 = \text{Span}\{v_1, v_2, v_3\}$

& Recall also that v_1, v_2 & v_3 are linearly independent. Any collection of 3 vectors $\{w_1, w_2, w_3\}$ s.t.

(i) $\mathbb{R}^3 = \text{Span}\{w_1, w_2, w_3\}$ &

(ii) $\{w_1, w_2, w_3\}$ are linearly independent is called a Basis for $\mathbb{R}^3$ (a set of axes geometrically for $\mathbb{R}^3$)

<u>Ex</u>: Prove that $(1,2,1)$, $(1,3,1)$ & $(1,0,1)$ is not a Basis

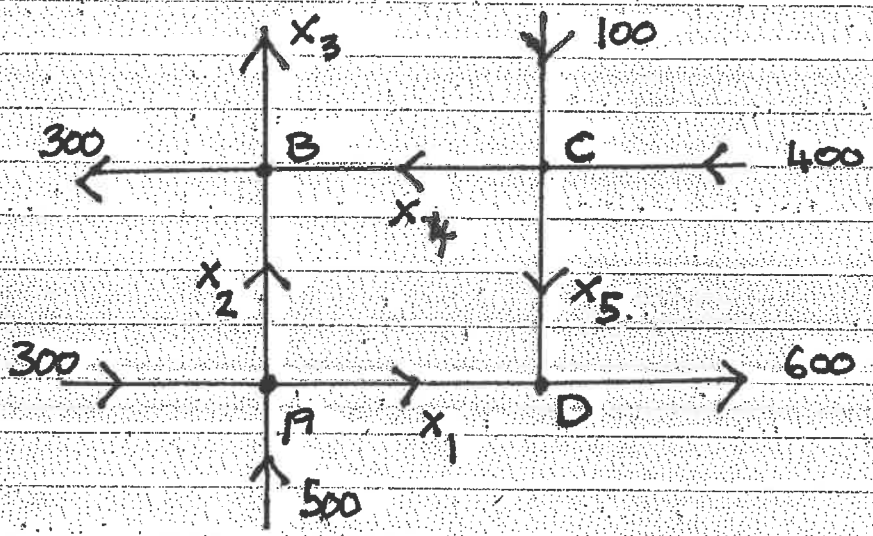
Hint: Can you express $(1,1,-2)$ as a linear combination of $(1,2,1)$, $(1,3,1)$ & $(1,0,1)$?

Applications: Systems of linear eqns arise naturally in many area of Science, Business & Maths & Computing

eg. Balancing chemical eqns &

Studying Network Flow.

Ex: Consider The ~~traffic~~ flow in a City Illustrated By the following diagram.



At each Junction A, B, C, D flow in = flow out.

	Flow in		Flow out
A	$300 + 500$	=	$x_1 + x_2$
B	$x_2 + x_4$	=	$300 + x_3$
C	$100 + 400$	=	$x_4 + x_5$
D	$x_1 + x_5$	=	600

& Total flow in = Total flow out

$$\Leftrightarrow \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 800 \\ 0 & 1 & -1 & 1 & 0 & 300 \\ 0 & 0 & 0 & 1 & 1 & 500 \\ 1 & 0 & 0 & 0 & 1 & 600 \\ 0 & 0 & 1 & 0 & 0 & 400 \end{array}\right)$$

The last eqn comes from Total flow

in = total flow out ie

$$500 + 300 + 100 + 400 = 300 + x_3 + 600.$$

Solving we see that x_5 is free

$x_1 = 600 - x_5$, $x_2 = 200 + x_5$, $x_3 = 400$,

$x_4 = 500 - x_5$ & x_5 free.