§ Linear independence of vectors in $\mathbb{R}^n$.

We have seen examples of systems of linear eqns which contain redundant equations in the sense that some of the equations may be obtained from others, i.e. in the sense that some of the equations can be written as a linear combination of others. Equivalently given a collection of vectors in $\mathbb{R}^n$ can we write some of them as linear combination of the others

If not we say that the collection is linearly independent (& If NOT linearly dependent).

Ex: Is the collection $v_1 = (1,2,1)$, $v_2 = (3,-1,2)$, $v_3 = (9,4,7)$ linearly independent?

<u>Ans</u>: No they the vectors are linearly dependent as

$$(9,4,7) = 3(1,2,1) + 2(3,-1,2)$$

$$\Longleftrightarrow \quad 1(9,4,7) - 3(1,2,1) - 2(3,-1,2) = (0,0,0)$$

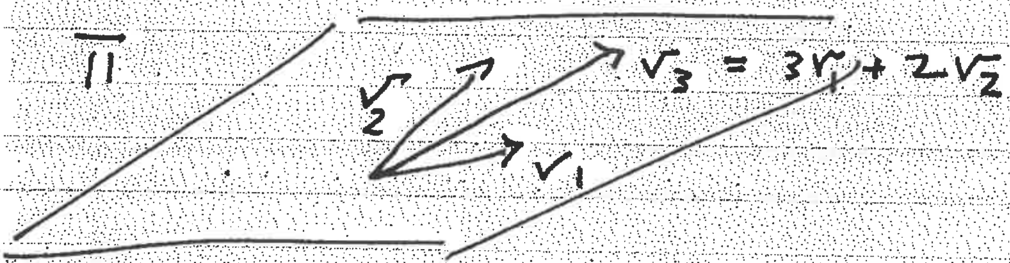
ie $\exists$'s $x_1, x_2, x_3 \in \mathbb{R}$ not all zero

s.t $x_1(9,4,7) + x_2(1,2,1) + x_3(3,-1,2)$

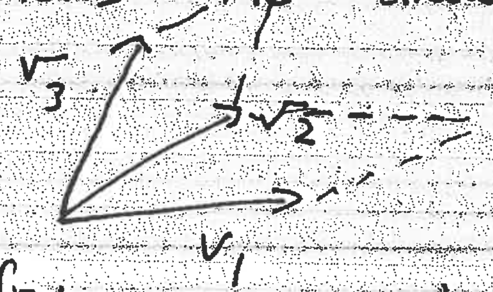
$= (0,0,0)$

(in this case $x_1 = 1$, $x_2 = -3$ & $x_3 = -2$.)

Geometrically: $v_3 = (9,4,7)$ lies in the plane $\Pi := \{ v \in \mathbb{R}^3 \mid v = sv_1 + tv_2,\ s,t \in \mathbb{R}$



If v_1, v_2 & v_3 were linearly independent they would not lie in a plane & would need 3 dimensions to draw them e.g.



They form the sides of a box

Formal Defn:
A collection of vectors $\{v_1, .., v_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if (the vector equation)

$$x_1 v_1 + x_2 v_2 + \cdots x_p v_p = 0 = (0,0,..,0)$$

has only the trivial solution

$$x_1 = x_2 = \cdots = x_p = 0 \in \mathbb{R}.$$

and the collection $\{v_1, \ldots, v_p\}$ of vectors in $\mathbb{R}^n$ is said to be linearly dependent if $\exists$'s real numbers $c_1, c_2, \ldots, c_p$ not all zero s.t. $c_1 v_1 + c_2 v_2 + \cdots + c_p v_p = 0 \in \mathbb{R}^n$

(ie the vector eqn $x_1 v_1 + x_2 v_2 + \cdots x_p v_p = 0 \in \mathbb{R}^n$ has a non trivial soln.).

Ex: Let $v_1 = (1,2,3)$, $v_2 = (4,5,6)$ & $v_3 = (2,1,0)$. Determine if the set $\{v_1, v_2, v_3\}$ is linearly independent.

ie Can we find x_1, x_2, x_3 not all zero s.t. $x_1(1,2,3) + x_2(4,5,6) + x_3(2,1,0) = (0,0,0)$

ie

$$\begin{aligned} x_1 + 4x_2 + 2x_3 &= 0 \\ 2x_1 + 5x_2 + x_3 &= 0 \\ 3x_1 + 6x_2 \quad\quad &= 0 \end{aligned} \iff$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 2 & 5 & 1 & 0 \\ 3 & 6 & 0 & 0 \end{array}\right) \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{array}\right) \xrightarrow{R_3 - 2R_2} \left(\begin{array}{ccc|c} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So x_1 & x_2 are non-free (or basic) variables & x_3 'n free (ie free to take any Real number $t \in \mathbb{R}$ as its value)

& Then x_1 & x_2 are determined by

$$-3x_2 - 3x_3 = 0 \Rightarrow x_2 = -x_3 = t$$

& $$x_1 + 4x_2 + 2x_3 = 0 \Rightarrow x_1 = -4x_2 - 2x_3$$

$$\Rightarrow x_1 = -4(-t) - 2t = 2t$$

All Soln are $\{(2t, -t, t) \in \mathbb{R}^3 \mid t \in \mathbb{R}\}$

in particular can choose $t \neq 0$, eg $t = 1$ to get:

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 1$$

So $$2v_1 - v_2 + v_3 = 0$$ so

v_1, v_2 & v_3 are not linearly independent ie are linearly dependent as we have found x_1, x_2 & x_3 not all zero with $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$

Alternatively $v_3 = -2v_1 + v_2$

ie v_3 lies in the span of v_1 & v_2 as we now define.