Defn: We say that a system of linear eqns is consistent if it has any solutions ie either a unique soln OR infinitely many & inconsistent otherwise, ie has no soln.

Ex The above system has a unique soln, so is consistent.

Ex: Find the solns if any of

$$\begin{aligned} x_2 - 4x_3 &= 8 \\ 2x_1 - 3x_2 + 2x_3 &= 1 \\ 5x_1 - 8x_2 + 7x_3 &= 1 \end{aligned} \quad \longleftrightarrow$$

$$\left(\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 5 & -8 & 7 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2}$$

Pivot

$$\left(\begin{array}{ccc|c} \boxed{2} & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 5 & -8 & 7 & 1 \end{array}\right) \xrightarrow{R_3 - \frac{5}{2}R_1}$$

$$\left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & \boxed{1} & -4 & 8 \\ 0 & -\frac{1}{2} & 2 & -\frac{3}{2} \end{array}\right) \xrightarrow{R_3 + \frac{1}{2}R_2}$$

$$\left(\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{array}\right)$$

But the last row of this augmented matrix says that

$$0x_1 + 0x_2 + 0x_3 = 5/2 \quad ↯$$

So there are no soln, The system is inconsistent.

Echelon form & Reduced Echelon form:

Recall in solving The System of eqns with augmented matrix

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array}\right)$$

we used elementary Row operations to Put it in Echelon form

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{array}\right)$$

& used "Back substitution" to get $x_1 = 10$, $x_2 = 20$ & $x_3 = 10$

Alternatively we could have continued & Placed as many zeros as possible above the Pivots in each column as follows to Put the matrix (augmented) in "Reduced Echelon form":

$$\left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{array}\right) \xrightarrow{R_3 \to R_3/130} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & \boxed{1} & -6 & -40 \\ 0 & 0 & 1 & 10 \end{array}\right)$$

$$\xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & 10 & 110 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & \boxed{1} & 10 \end{array}\right) \xrightarrow[R_2 + 6R_3]{R_1 - 10R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 10 \end{array}\right) \Rightarrow x_1 = 10,\ x_2 = 20 \ \& \ x_3 = 10.$$

THE REDUCED ECHELON FORM is useful when looking for all Solns ie the "general Soln" of a system with infinitely many Solns:

Ex: Find the general Soln of the system whose augmented matrix has been reduced to:

$$\left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right)$$

Surprise! the augmented matrix is already in Echelon form (the 1st non zero element (Pivot) in each Row is to the right of the row below it). Now put it in Reduced Echelon form (use Pivots to put zero's above them)

$$\left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right) \xrightarrow[R_1+2R_3]{R_2+R_3} \left(\begin{array}{ccccc|c} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right)$$

$$\xrightarrow{R_1-R_2} \left(\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right) \xrightarrow{R_2 \to \frac{1}{2}R_2} \left(\begin{array}{ccccc|c} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{array}\right)$$

So
$$\begin{aligned} x_1 + 6x_2 + 3x_4 &= 0 \\ x_3 - 4x_4 &= 5 \\ x_5 &= 7 \end{aligned}$$

The Columns with a Pivot (a 1st non zero in a Row) are Col 1, Col 3 & Col 5 $\longleftrightarrow$ to (non-free variables x_1, x_3, x_5

The remaining variables x_2 & x_4 are called free variables as they can or are free to

Be any Real numbers & the non free variables are determined in terms of the free ones as follows:

$$x_1 = -6x_2 - 3x_4$$

x_2 is free ie $x_2 = s \in \mathbb{R}$ (any s)

$$x_3 = 5 + 4x_4$$

x_4 is free ie $x_4 = t \in \mathbb{R}$ (any t)

& $x_5 = 7$.

So the "solution space" is the set of all solutions is

$$S := \{ (-6s-3t, s, 5+4t, t, 7) \in \mathbb{R}^5 \mid s,t \in \mathbb{R} \}$$

Ex: take special values for s & t e.g. $s = 3$, $t = 2$ & verify that e.g. $(-24, 3, 13, 2, 7)$ does indeed satisfy the three original eqns.

OR $s = 0$, $t = 0$ so that $(0,0,5,0,7)$ is a solution of the original $Ax = b$

NOTE: The set of all solutions S can be also written as

$S = \{ (-6s-3t, s, 4t, t, 0) + (0,0,5,0,7) \in \mathbb{R}^5 \mid s,t \in \mathbb{R} \}$

and (Check)

$S' = \{ (-6s-3t, s, 4t, t, 0) \in \mathbb{R}^5 \mid s,t \in \mathbb{R} \}$

is the set of all solutions of the system

$$\underset{3\times5\ 5\times1}{AX} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}_{=3\times1} \quad \left(A = \begin{pmatrix} 1 & 6 & 2 & -5 & -2 \\ 0 & 0 & 2 & -8 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \right.$$

Called the corresponding Homogeneous system & $P = (0,0,5,0,7)$ is one "Particular" soln of $AX = b$ (check)

This is a General fact for systems of (Consistent) linear eqns. ie the solns of $AX = b$ are all of the form $X' + P$ where X' is a soln of $AX = 0$ & P is one "particular" soln of $AX = b$.

Ex: Solve the Homogeneous System of eqns

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 0 \\ 2x_1 - 3x_2 + 4x_3 - 3x_4 &= 0 \\ 3x_1 - 5x_2 + 5x_3 - 4x_4 &= 0 \\ -x_1 + x_2 - 3x_3 + 2x_4 &= 0 \end{aligned}$$

The augmented matrix is :

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 2 & -3 & 4 & -3 & 0 \\ 3 & -5 & 5 & -4 & 0 \\ -1 & 1 & -3 & 2 & 0 \end{array}\right) \xrightarrow[\substack{R_3 - 3R_1 \\ R_4 + R_1}]{R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[R_4 + R_2]{R_3 - R_2} \left(\begin{array}{cccc|c} 1 & -2 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1 + 2R_2}$$

(ECHELON FORM)

$$\left(\begin{array}{cccc|c} 1 & 0 & 5 & -3 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$ (Reduced ECHELON FORM)

So The Cols with a (non zero) Pivot are Col 1 & Col 2 so x_1 & x_2 are non-free (or Basic Variables) & the others ie x_3 & x_4 are free variable ie they can be any Real numbers r & s. So the general soln (or the set of all solution S

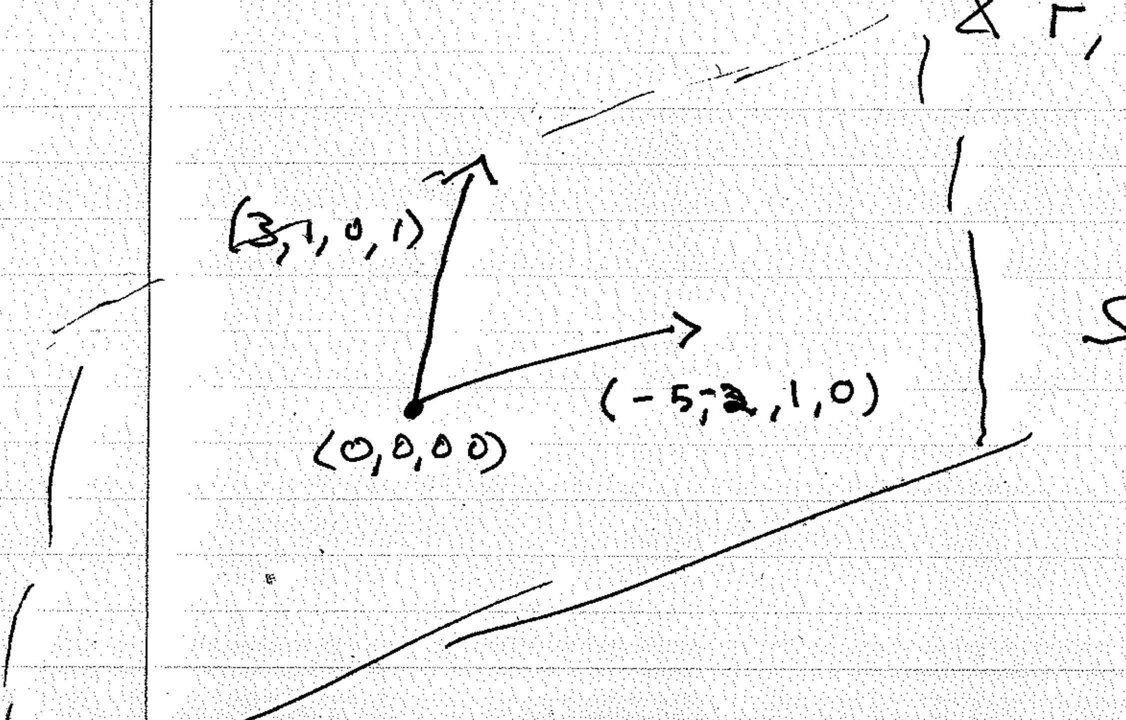
is
$$x_1 = -5x_3 + 3x_4$$
$$x_2 = -2x_3 + x_4$$
$$x_3 = r$$
$$x_4 = s$$

ie $S = \{(-5r + 3s, -2r + s, r, s) \in \mathbb{R}^4 \mid r, s \in \mathbb{R}\}$

Geometrically, S is a 2-dimensional plane in $\mathbb{R}^4$ through the origin $(0,0,0,0)$ Because if $x \in S$ then $x = (-5r+3s, -2r+s, r, s)$

$$= r(-5,-2,1,0) + s(3,1,0,1)$$

$\& \ r, s \in \mathbb{R}$



Note letting $r = s = 0$ we see that $(0,0,0,0) \in \mathbb{R}$.

Ex