

§ Solving Systems of Linear Equations and ECHELON FORM:

Some systems are easily solved e.g

$$x_1 + 2x_2 + x_3 = 16 \quad \text{--- (i)}$$

$$x_2 + 2x_3 = 10 \quad \text{--- (ii)}$$

$$3x_3 = 9 \quad \text{--- (iii)}$$

This is because of the "upper diagonal" shape of the eqns, i.e. eqn (ii) involves one less unknown than eqn (i) & eqn (iii) involves one less unknown again than eqn (ii).

Soln: A solution to the above system

of linear eqns (i.e. more than one

eqn & the unknowns $x_1, x_2, & x_3$ can only be multiplied by real

numbers & then added (or subtracted)

is just a vector (c_1, c_2, c_3) so that if

$$x_1 = c_1, \quad x_2 = c_2, \quad x_3 = c_3$$

then all three eqns (i), (ii) & (iii) are all satisfied. In this case

(iii) $\Rightarrow x_3 = 3$. Back substituting

this value into (ii) $\Rightarrow x_2 + 2(3) = 10$

$\Rightarrow x_2 = 4$ & Finally back substituting

In (i) $\Rightarrow x_1 + 2(4) + 3 = 16$

$\Rightarrow x_1 = 5$

So in this case we have a unique
Soln $x_1 = 5$, $x_2 = 4$ & $x_3 = 3$

OR Simply $(5, 4, 3)$.

Aside: We will also use the definition of
Matrix multiplication to write a system
of linear eqns as one matrix eqn.

ABOVE Let $A := \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ 3×3

& $X := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ & $b := \begin{pmatrix} 16 \\ 10 \\ 9 \end{pmatrix}$
a 3×1 matrix a 3×1 matrix

Then our eqns can be written as:

$$AX = b$$

$$\underline{3 \times 3} \underline{3 \times 1} = 3 \times 1$$

IDEA: CHANGE A GIVEN SYSTEM OF
LINEAR EQNS ~~TO~~ A SYSTEM THAT
HAS A "SIMPLE" FORM AS ABOVE

AND THAT HAS THE SAME SOLUTIONS.
i.e. TO A SO CALLED "EQUIVALENT SYSTEM"

Question: WHAT CAN WE DO TO A SYSTEM
OF LINEAR EQNS WITHOUT CHANGING
THE SOLNS?

Elementary operations:

- ANSWER: (1) Interchange 2 equations
- (2) Multiply an equation by a non zero real number
- (3) Replace an equation by itself plus a multiple (non-zero) of some other eqn in the system.

When we write the system of eqns in "extended matrix notation" i.e.:

$$(i) \dots a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$(ii) \dots a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

\vdots

$$(m) \dots a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m$$

REWRITE
AS

$AX = b$ OR simply encode this info
OR Augmented

in the extended matrix $(A|b)$

$$\left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array} \right)$$

So that equation

- eqn (i) corresponds to Row 1 of $(A|b)$
- eqn (ii) " " Row 2 of $(A|b)$
- \vdots
- eqn (m) " " Row m of $(A|b)$

Letting R_i denote Row i of $(A|b)$

Then the Elementary (Row) operations:

are written as:

(1) Interchange R_i & R_j

(2) Replace R_i by kR_i , $k \neq 0$, $k \in \mathbb{R}$

(3) Replace R_i by $R_i + kR_j$, $j \neq i$, $k \in \mathbb{R}$

Ex: Consider the system of 3 linear eqns in 3 unknowns $x_1, x_2 \Delta x_3$:

$$2x_1 + 3x_2 + 2x_3 = 100$$

$$x_1 + x_2 + 4x_3 = 70$$

$$20x_1 + 10x_2 + 10x_3 = 500$$

$$\leftrightarrow \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 4 \\ 20 & 10 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 500 \end{pmatrix}$$

keep track in the augmented matrix

\leftrightarrow

$$\begin{pmatrix} 2 & 3 & 2 & | & 100 \\ 1 & 1 & 4 & | & 70 \\ 20 & 10 & 10 & | & 500 \end{pmatrix} := (A|b)$$

Now use The Elementary Row operations so that $(A|b)$ is in so called "Echelon form" i.e. "upper triangular" form like the 1st "simple" eqns we solved above

The idea is to use the first non-zero entry in a Row (Pivot) to place zeros in the Column entries Below it. So:

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 4 & 70 \\ 2 & 3 & 2 & 100 \\ 20 & 10 & 10 & 500 \end{array} \right)$$

Pivot

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 - 20R_1 \end{array} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & \boxed{1} & -6 & -40 \\ 0 & -10 & -70 & -900 \end{array} \right) \xrightarrow{R_3 + 10R_2} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{array} \right)$$

So Row 3 says $-130x_3 = -1300$
 $\Rightarrow x_3 = 10$

Row 2 says $x_2 - 6x_3 = -40$ i.e.
 $x_2 - 60 = -40 \Rightarrow x_2 = 20$

& Row 1 says $x_1 + x_2 + 4x_3 = 70$
 $\Rightarrow x_1 + 20 + 40 = 70$
 $\Rightarrow x_1 = 10$