§ Solving Systems of Linear equations and ECHELON FORM:

Some Systems are easily Solved e.g

$$x_1 + 2x_2 + x_3 = 16 \quad \cdots (i)$$
$$x_2 + 2x_3 = 10 \quad \cdots (ii)$$
$$3x_3 = 9 \quad \cdots (iii)$$

This is Because of the "upper diagonal" Shape of the eqns, ie eqn (ii) involves one less unknown than eqn (i) & eqn (iii) involves one less unknown again than eqn (ii).

Soln: A Solution to the above system of Linear eqns (i.e MORE than one eqn & the unknowns $x_1, x_2, \& x_3$ can only Be multiplied By Reals numbers & then added (OR subtracted)) is just a vector (c_1, c_2, c_3) so that if $x_1 = c_1$, $x_2 = c_2$, $x_3 = c_3$ then all three eqns (i), (ii) & (iii) are all Satisfied. In this case

(iii) $\Rightarrow x_3 = 3$. Back substituting this value into (ii) $\Rightarrow x_2 + 2(3) = 10$ $\Rightarrow x_2 = 4$ & Finally Back substituting In (i) $\Rightarrow$ $x_1 + 2(4) + 3 = 16$

$$\Rightarrow \quad x_1 = 5$$

So in this case we have a unique Soln $x_1 = 5$, $x_2 = 4$ & $x_3 = 3$

OR Simply $(5, 4, 3)$.

<u>Aside:</u> We will also use the definition of Matrix multiplication to write a System of Linear eqns as one matrix eqn.

ABOVE Let $A := \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ 3x3

& $X := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ & $b := \begin{pmatrix} 16 \\ 10 \\ 9 \end{pmatrix}$

a 3x1 matrix a 3x1 Matrix

Then our eqns can be written as:

$$\underset{3\times 3 \; 3\times 1 \; = \; 3\times 1}{Ax = b}$$

IDEA: CHANGE A GIVEN SYSTEM of Linear eqns ~~INTO~~ A SYSTEM that has a "simple" form as ABOVE

AND THAT HAS THE SAME Solutions.

ie To A So Called "equivalent system"

<u>Question</u>: WHAT CAN WE DO to A system of Linear eqns WITHOUT CHANGING THE Solns?

Elementary operations:

ANSWER: (1) Interchange 2 equations

(2) Multiply an equation By a non zero Real number

(3) Replace an equation By itself plus a multiple (non-zero) of some other eqn in the system.

When we write the system of eqns in "Extended matrix notation ie:

(i) ... $a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

(ii) ... $a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$ ReWRite as

$\vdots$

(m) ... $a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$

$AX = b$ OR simply encode this info in the extended or Augmented Matrix

$$(A|b) \quad \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & & \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

So that equation

(i) corresponds to Row 1 of $(A|b)$

eqn (ii) " " Row 2 of $(A|b)$

$\vdots$

eqn (m) " " Row m of $(A|b)$

Letting R_i denote Row i of $(A|b)$

Then the Elementary (Row) operations are written as:

(1) Interchange R_i & R_j

(2) Replace R_i By kR_i, $k \neq 0$, $k \in \mathbb{R}$

(3) Replace R_i by $R_i + kR_j$, $j \neq i$, $k \in \mathbb{R}$

Ex: Consider the system of 3 linear eqns in 3 unknowns x_1, x_2 & x_3:

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &= 100 \\ x_1 + x_2 + 4x_3 &= 70 \\ 20x_1 + 10x_2 + 10x_3 &= 500 \end{aligned} \iff \begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 4 \\ 20 & 10 & 10 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 500 \end{pmatrix}$$

keep track in the augmented matrix

$$\iff \left(\begin{array}{ccc|c} \multicolumn{3}{c|}{A} & b \\ 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array}\right) := (A|b)$$

Now use The Elementary Row operations so that $(A|b)$ is in so called "Echelon form" ie "upper triangular" form like the 1st "simple" eqns we solved above

THE idea is to use the first non-zero entry in a Row (Pivot) to place zeros in the Column entries Below it. So:

$$\left(\begin{array}{ccc|c} 2 & 3 & 2 & 100 \\ 1 & 1 & 4 & 70 \\ 20 & 10 & 10 & 500 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} \boxed{1} & 1 & 4 & 70 \\ 2 & 3 & 2 & 100 \\ 20 & 10 & 10 & 500 \end{array}\right)$$

Pivot

$$\xrightarrow[R_3 - 20R_1]{R_2 - 2R_1} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & \boxed{1} & -6 & -40 \\ 0 & -10 & -70 & -900 \end{array}\right) \xrightarrow{R_3 + 10R_2} \left(\begin{array}{ccc|c} 1 & 1 & 4 & 70 \\ 0 & 1 & -6 & -40 \\ 0 & 0 & -130 & -1300 \end{array}\right)$$

So Row 3 says $-130x_3 = -1300$

$\Rightarrow x_3 = 10$

Row 2 says $x_2 - 6x_3 = -40$ ie

$x_2 - 60 = -40 \Rightarrow x_2 = 20$

& Row 1 says $x_1 + x_2 + 4x_3 = 70$

$\Rightarrow x_1 + 20 + 40 = 70$

$\Rightarrow x_1 = 10.$