§ Applications of Eigenvalues & eigenvectors.

Markov Processes: Recall that a Markov Process consists of a Population a discrete collection of states (2 last year & 3 this year) in one or other of which every individual in the population must be at a given time. At discrete fixed time intervals individuals can change to a different state or not & we are given the proportions of the population that change between any 2 states (or not).

Example: Summer 2015-16 Q. 4.

The City of Markoville has 3 grocery stores, ERO, Singular & Eigenvalu. Every resident of Markoville is a customer of exactly one of these stores

- Of those who do their grocery shopping with ERO at a particular time 70% will still be with ERO six months later, 10% will have moved to Singular & 20% will have moved to Eigenvalu.

- Of those who shop with Singular at a given time, 50% will still be with Singular six months later, 30% will have moved to ERO & 20% will have moved to Eigenvalu.

• Of those residents who do their grocery shopping with Eigenvalu at a particular time, 40% will still be with Eigenvalu six months later, 30% will have moved to ERO & 30% will have moved to Singular.

(a) Write down the transition matrix for this Markov process & explain what it does.

We will denote by x_0, y_0 & z_0 the proportions of Markoville residents shopping in ERO, Singular & Eigenvalu respectively at time 0 & similarly x_1, y_1 & z_1 the proportions in ERO, Singular & Eigenvalue at one time interval later (ie 6 months later). At 2 time intervals later (ie 2 x 6 months) we denote the proportions shopping at ERO, Singular & Eigenvalu by x_2, y_2 & z_2 respectively ... & n time intervals later (6n months) by x_n, y_n & z_n.

Then

$$x_1 = .7x_0 + .3y_0 + .3z_0$$
$$y_1 = .1x_0 + .5y_0 + .3z_0$$
$$z_1 = .2x_0 + .2y_0 + .4z_0$$

(from the given information)

i.e. $\begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} = T \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$

where $T = \begin{pmatrix} \cdot 7 & \cdot 3 & \cdot 3 \\ \cdot 1 & \cdot 5 & \cdot 3 \\ \cdot 2 & \cdot 2 & \cdot 4 \end{pmatrix}$ is

The so called transition matrix for the process. T tells us the proportions in each state at time interval 1 by multiplying the proportions in each state at time interval 0 by T on the front.

Aside: Some Notation $X^{(0)} := \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$, $X^{(1)} := \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}$

$X^{(2)} := \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$ ... $X^{(n)} := \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$

So $X^{(1)} = TX^{(0)}$, $X^{(2)} = TX^{(1)} = TTX^{(0)}$
$= T^2 X^{(0)}$

$X^{(3)} = TX^{(2)} = TT^2X^{(0)} = T^3X^{(0)}$
. . . . etc
& $X^{(n)} = TX^{(n-1)} \cdots = T^n X^{(0)}$.

So we know the proportions in the various states ie x_n, y_n & z_n if we know T^n & x_0, y_0 & z_0.

We have just seen how to calculate T^n using the eigenvalues & eigenvector of T.

Often we are only interested in the long term (steady state) proportions $X^{(n)}$ when $n \to \infty$ (n very large)
It is the case that as $n \to \infty$ the proportions in each state ie $X^{(n)} = \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix}$ don't change, ie

$$X^{(n)} \approx X^{(n-1)} \approx X \quad \text{(some fixed proportions)}$$

$$\Rightarrow \quad X = X^{(n)} = T X^{(n-1)} = TX \quad (\text{as } n \to \infty)$$

ie $TX = X \Rightarrow T$ has an eigen-value $\lambda = 1$. This is always the case for a Markov Process.

Back to Q.4. Summer 2015-16 Part (b)

Explain why the Transition Matrix T has an eigen value $\lambda = 1$.

<u>Answer</u>: T has an e-value $\lambda = 1 \iff \det(T - 1I) = 0 \iff (T - 1I)$ has no inverse. Now

$$(T - I) = \begin{pmatrix} -0.3 & 0.3 & 0.3 \\ 0.1 & -0.5 & 0.3 \\ 0.2 & 0.2 & -0.4 \end{pmatrix} \longrightarrow$$

$$\xrightarrow{R_1 \to R_1 + R_2 + R_3} \begin{pmatrix} 0 & 0 & 0 \\ 0.1 & -0.5 & 0.3 \\ 0.2 & 0.2 & -0.6 \end{pmatrix}$$ so the $\text{rank}(T - 1I) = 2 < 3$

$\Rightarrow (T - 1I)$ has no inverse $\Leftrightarrow \lambda = 1$ an eigenvalue of T.

Part (c) Singular is a relative newcomer to Markoville & its directors aim in the long term to maintain a $1/3$ share of the local grocery market. Today 50% of Markoville residents are customers of ERO, 20% are customers of Singular & 30% are customers of Eigenvalue. If the current trends continue, can they expect to achieve this aim?

Answer: We need to find the steady state proportions in each of the three states ERO, Singular & Eigenvalue (its independent of x_0, y_0 & z_0!)

ie Find the eigenvector $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ of T

Corresponding to $\lambda = 1$ s.t. $x + y + z = 1$ (Because the proportions in the steady state x, y & z must add to 1 = Entire population).

ie find $X \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ s.t.

$$(T - 1I)X = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

ie Solve $\left(\begin{array}{ccc|c} -0.3 & 0.3 & 0.3 & 0 \\ 0.1 & -0.5 & 0.3 & 0 \\ 0.2 & 0.2 & -0.6 & 0 \end{array}\right)$

multiplying all eqns By 10 $\Leftrightarrow$ $\left(\begin{array}{ccc|c} -3 & 3 & 3 & 0 \\ 1 & -5 & 3 & 0 \\ 2 & 2 & -6 & 0 \end{array}\right) \xleftrightarrow{R_1 \leftrightarrow R_2}$

$\left(\begin{array}{ccc|c} 1 & -5 & 3 & 0 \\ -3 & 3 & 3 & 0 \\ 2 & 2 & -6 & 0 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 + 3R_1}$

$\left(\begin{array}{ccc|c} 1 & -5 & 3 & 0 \\ 0 & -12 & 12 & 0 \\ 0 & 12 & -12 & 0 \end{array}\right)$

$\Rightarrow \quad 12y = 12z \Rightarrow y = z$

$\& \quad x = 5y - 3z \Rightarrow x = 2z$

$\therefore X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ z \\ z \end{pmatrix}$ & want

$2z + z + z = 1 \Rightarrow z = 1/4$

$\Rightarrow$ The Steady state is $X = \begin{pmatrix} 1/2 \\ 1/4 \\ 1/4 \end{pmatrix}$ Singlu

Since $1/4 < 1/3$. Singular will not achieve their goal.

To show T has an e-value of T $\lambda = 1$.

Alternatively: The columns all sum to 1 $\Rightarrow$ If

$X = (1, 1, 1)$ Then.

$$XT = (1,1,1)\begin{pmatrix} .7 & .3 & .3 \\ .1 & .5 & .3 \\ .2 & .2 & .4 \end{pmatrix} = (1\ 1\ 1) = X$$

$\Rightarrow$ $(XT)^T = X^T$

$\Rightarrow$ $T^T X^T = X^T$ $\Rightarrow$ X^T is an eigenvector for T^T with e-value $\lambda = 1$. But T & T^T have the same eigenvalues as we saw because $P_{T^T}(\lambda) = |T^T - \lambda I|$

$= |T^T - \lambda I^T| = |(T - \lambda I)^T|$

$= |T - \lambda I|$.

($|B^T| = |B|$ for any B $n \times n$ matrix)