§ Eigenvalues & eigenvectors:

Recall that an $n \times n$ Matrix A give a linear map by placing a vector $X = (x_1, \dots, x_n)$ as a column or $n \times 1$ matrix $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ behind A & multiplying to get

$$\underset{n \times n}{A} \underset{n \times 1}{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} = \underset{n \times 1}{\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}} \longleftrightarrow (y_1, \dots, y_n) \in \mathbb{R}^n.$$

We also saw that a linear map sends lines through the origin in $\mathbb{R}^n$ to a line through the origin. So ask are there any such lines fixed by A.

So we want line through the origin $\ell = \{ tv \mid t \in \mathbb{R},\ v \in \mathbb{R}^n \}$, $v \neq (0 0 \cdots 0)$

with $A\ell = \ell$ (As $A(0) = 0$)

$$\Rightarrow \quad A(tv) = sv, \quad t \neq 0, \quad t, s \in \mathbb{R}$$

$$\Rightarrow \quad tA(v) = sv$$

$$\Rightarrow \quad A(v) = \left(\frac{s}{t}\right) v = \lambda v \qquad \lambda \in \mathbb{R},$$

& $v \neq 0$ vector

$$\Rightarrow \quad Av = \lambda v = \lambda I v$$

$$* \quad \Rightarrow \quad (A - \lambda I) v = 0 \quad \text{(zero vector)}$$

$$\Rightarrow \quad (A - \lambda I) \text{ is not invertible}$$

as if $(A-\lambda I)^{-1}$ existed we can multiply * on Both sides By it to get $v = 0$ But we assumed that $v \neq 0$ (as it has to define a direction of a line through the origin)

Therefore $\det(A - \lambda I) = 0$ **

This is a Polynomial in λ of degree n.

and its roots make ** Satisfied

Ex: $n=3$ Let $A = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}$

$$A - \lambda I = \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

$$= \begin{pmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{pmatrix}$$

If λ is an eigenvalue of A we saw that $\det(A - \lambda I) = 0$

We can ~~e~~valuate this straight off OR can simplify thing By using row operations

& The theorem above that tells us how The determinant changes

$$\xrightarrow{R_1+R_2} \begin{pmatrix} 2-\lambda & 2-\lambda & 0 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{pmatrix}$$

(· This has the same det as $A-\lambda I$ By Thm part (a) which

$$= (2-\lambda) \det \begin{pmatrix} 1 & 1 & 0 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{pmatrix}$$

By Part (c)) So $\det(A-\lambda I)$

$$= (2-\lambda)\left\{(-\lambda)(5-\lambda)+4-\left[(5-\lambda)-4\right]\right\}$$
$$= (2-\lambda)\left\{\lambda^2-4\lambda+3\right\}$$
$$= (2-\lambda)(\lambda-1)(\lambda-3)$$

So $\det(A-\lambda I)=0 \iff \lambda = 1, 2, 3$

Now for each eigenvalue $\lambda = 1, 2, 3$ we solve $Av = \lambda v \iff (A-\lambda I)v = \begin{pmatrix}0\\0\\0\end{pmatrix}$ for the corresponding eigenvector v.

Let $v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

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$$(A - I)\,v = 0$$
$$\shortparallel$$
$\lambda = 1.$ $$(A - \lambda I)\,v = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff$$

$$\begin{pmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \quad 2y - z = 0 \qquad (i)$$
$$\& \quad x - y + z = 0 \qquad (ii)$$
$$(\quad 4x - 4y + 4z = 0 \quad \text{same eqn})$$

$(i) \Rightarrow z = 2y$ $\;\&\; (ii) \Rightarrow x = y - 2y$
$= -y$

So $v = \begin{pmatrix} -y \\ y \\ 2y \end{pmatrix}$ (a line of solns as we expect as generically 2 planes intersect in a line)

e.g. $y = 1$ gives us an eigenvector $v = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ with corresponding eigen vector $\lambda = 1$.

(Check: $Av = 1v = v$

$$\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \quad \checkmark$$)

Find an eigenvector for $\lambda = 2$.

ie Solve $(A - \lambda I)\,v = 0$ ie

$$(A - 2I)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

ie $\begin{pmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(i) $\Rightarrow$ $z = -x + 2y$ (Note (ii) $\Leftrightarrow$ (i))

(iii) $\Rightarrow$ $4x - 4y + 3(2y - x) = 0$

$\Rightarrow$ $x = -2y$

So $v = \begin{pmatrix} -2y \\ y \\ 4y \end{pmatrix}$ e.g. $y = 1$

$\Rightarrow$ $v = \begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix}$ is an eigenvector with eigenvalue $\lambda = 2$

Check:

Similarly get $v = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ is an eigenvector with $\lambda = 3$.

Check: $\begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 4 & -4 & 5 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 12 \end{pmatrix} = 3\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$

So indeed $A\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} = 3\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ ($Av = \lambda v$)

so that $\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}$ is an eigenvector of A with eigenvalue $\lambda = 3$

Now define a diagonal matrix D as:

$$D := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

& the matrix of corresponding eigen vectors E by

$$E := \begin{pmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{pmatrix}$$

& clearly

$$AE = \left(1\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} \; 2\begin{pmatrix} -2 \\ 1 \\ 4 \end{pmatrix} \; 3\begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} \right)$$

$$= \begin{pmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 2 & 4 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \implies$$

$$\boxed{AE = ED} \implies A = EDE^{-1}$$

& for any $n \in \mathbb{N}$ we have

$$A^n = \underbrace{EDE^{-1}EDE^{-1}EDE^{-1} \dots E^{-1}DE}_{n \text{ times}}$$

$$= ED^nE^{-1}$$ which we can easily calculate as $D^n = \begin{pmatrix} 1^n & 0 & 0 \\ 0 & 2^n & 0 \\ 0 & 0 & 3^n \end{pmatrix}$

(Because D is a diagonal matrix)

NOTE: The Reason that E^{-1} exists is that the 3 eigenvectors are linearly independent and this in turn is because all 3 eigenvalues are distinct

If we have a repeated eigenvalue then we may or may not be able to get 3 linearly indep. eigenvectors.

Eg. Let $A = \begin{pmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{pmatrix}$

Then $P(\lambda) = \det(A - \lambda I) = (\lambda+2)^2(\lambda+3)$

ie $\lambda = -2, -2, \& -3$ are the eigenvalues

$\lambda = -2$: To find the eigenvector or 2 of them we solve $(A - (-2)I)v = 0$ ie solve

$$\begin{pmatrix} 2 & -6 & -4 \\ 5 & -9 & -6 \\ -6 & 9 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2t \\ 3t \end{pmatrix} \quad \text{eg } t = 1 \quad \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$$

We just get one eigenvector for $\lambda = -2$ & for $\lambda = 3$ get an eigenvector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

So only 2 eigenvectors. Even still the Matrix $E = \begin{pmatrix} 0 & 0 & 2 \\ 0 & -2 & -1 \\ 0 & 3 & 3 \end{pmatrix}$ will be such that if

$$D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \text{ then}$$

$AE = ED$ But E^{-1} doesn't exist.

However the matrix $A = \begin{pmatrix} 4 & 8 & -2 \\ -3 & -6 & 1 \\ 9 & 12 & -5 \end{pmatrix}$

again $P(\lambda) = (\lambda+2)^2(\lambda+3)$ so that $\lambda = -2, -2, -3$ are eigenvalues

For $\lambda = -3$ we get an eigenvector $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ (check) For

$\lambda = -2$ solve $\begin{pmatrix} 6 & 8 & -2 \\ -3 & -4 & 1 \\ 9 & 12 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(only one eqn) $\Rightarrow 3x + 4y - z = 0$

ie a Plane of solutions

$\Rightarrow z = 3x + 4y$, so any vector of the form $\begin{pmatrix} x \\ y \\ 3x+4y \end{pmatrix}$ will be an eigenvector with eigenvalue $\lambda = -2$

eg $\begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$ and. (clearly linearly indep.)

$$E = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 4 & 3 & 3 \end{pmatrix}$$ will be invertible

with $AE = ED$ $\quad D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$

$\Rightarrow A = EDE^{-1}$

$\Rightarrow A^n = ED^nE^{-1}$.

Defn: We say that an $n \times n$ Matrix A is diagonalisable if $\exists$'s an invertible matrix E & a diagonal matrix D s.t.

$$\boxed{A = EDE^{-1}}$$