§ Inverses via determinants.

Recall: For a 2×2 Matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$\det A$ or $|A|$ or $\Delta := a_{11}a_{22} - a_{12}a_{21}$

& if $\neq 0$ $\quad A^{-1} = \dfrac{1}{|A|}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$

<u>Now</u>: If $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \rightsquigarrow$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{11}a_{21} & a_{11}a_{22} & a_{11}a_{23} \\ a_{11}a_{31} & a_{11}a_{32} & a_{11}a_{33} \end{pmatrix} \xrightarrow[R_3 - a_{31}R_1]{R_2 - a_{21}R_1}$$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22} - a_{21}a_{12} & a_{11}a_{23} - a_{21}a_{13} \\ 0 & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{33} - a_{31}a_{13} \end{pmatrix}$$

If A is invertible either the the $(2,2)$ entry or $(3,2)$ is $\neq 0$ (or both)

Suppose the $(2,2)$ entry is $\neq 0$. Now multiply Row 3 By $(a_{11}a_{22} - a_{21}a_{12})$ & then to this new Row 3 add

$-(a_{11}a_{32} - a_{12}a_{31})$ times Row 2 to g

$$\longrightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{11}a_{22}-a_{12}a_{21} & a_{11}a_{23}-a_{13}a_{21} \\ 0 & 0 & a_{11}|A| \end{pmatrix}$$

where $|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a$
$- a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$
which must be non zero if A^{-1} exists

Recall that for a 3×3 matrix A its determinant $\det A$ or $|A|$ we defined or given in terms of determinants of 2×2 matrices as

$$\det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_1$$

where A_{ij} is the 2×2 Matrix obtained from A by deleting the i th row & j th column.

We can similarly inductively defn the det of an $n\times n$ Matrix A

e.g. If A was a 4×4 Matrix define $\det A$ by:

$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} - a_{14} \det A_{14}$

where now the A_{ij} are 3×3 matrices

Now we can get the determinant of 4×4

So define the det of a 5×5 as

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} - a_{14} \det A_{14} + a_{15} \det A_{15}$$

In General:

Def: Let A be an $n \times n$ matrix $(n \geq 2)$

$$\det A := a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots (-1)^{1+n} a_{1n} \det A_{1n}$$

We can build the alternating sign into the following notation

$$C_{ij} := (-1)^{i+j} \det A_{ij}$$ & then

(Called the (i,j) cofactor

$$\det A = a_{11} C_{11} + a_{12} C_{12} + \quad + a_{1n} C_{1n}$$

(Called the cofactor expansion across the first row of A)

Thm: The determinant of an $n \times n$ matrix A can be computed a cofactor expansion across any row or down any

Column. The cofactor expansion across the ith Row is:

$$\det A = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}$$

& The Cofactor expansion down the jth Col is:

$$\det A = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}$$

Ex: $A = \begin{pmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{pmatrix}$

Because the 3rd row of A contains 2 zeros we will use the cofactor expansion across it.

$$\det A = \underset{\overset{\shortparallel}{0}}{a_{31}} C_{31} + a_{32} C_{32} + \underset{\overset{\shortparallel}{0}}{a_{33}} C_{33}$$

$$\therefore \det A = a_{32} C_{32} = (-2)(-1)^{3+2} A_{32}$$

$$= (-2)(-1) \begin{vmatrix} 1 & 0 \\ 2 & -1 \end{vmatrix} = -2$$

Alternatively the cofactor expansion down the 3rd Column gives

$$\det A = a_{13} C_{13} + a_{23} C_{23} + a_{33} C_{33}$$

$$= 0 + (-1)(-1)^{2+3} A_{23} + 0$$

$$= A_{23} = \begin{vmatrix} 1 & 5 \\ 0 & -2 \end{vmatrix} = -2$$

Compute the det A, where

$$A = \begin{pmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{pmatrix}.$$

Use the cofactor expansion down the first column

$$\Rightarrow \det A = 3 \cdot \begin{vmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{vmatrix} + 0 \cdot C_{21} + 0 \cdot C_{31} + 0 \cdot C_{41} + 0 \cdot C_{51}$$

$$= 3 \cdot 2 \begin{vmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{vmatrix} = 3 \cdot 2\,(-2) = -12$$

§ Row operations & Determinants:

Thm: Let A be a square Matrix

(a) If a multiple of one row of A is added to another row of A to produce a matrix B then:

$$\det B = \det A$$

(b) If 2 rows of A are interchanged to produce a matrix B then

$$\det B = -\det A$$

(c) If a row of A is multiplied by $k \in \mathbb{R}$ $(k \neq 0)$ to produce a matrix

B then:

$$\det B = k \cdot \det A.$$

Ex

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{pmatrix}$$

Then $\begin{vmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -4 & 2 \\ 0 & 0 & -5 \\ 0 & 3 & 2 \end{vmatrix}$

$(R_2 \longleftrightarrow R_3)$ $= - \begin{vmatrix} 1 & -4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & -5 \end{vmatrix} = -(1)(3)$

$= 15$

Ex $\begin{vmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{vmatrix}$

$= 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{vmatrix} = 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{vmatrix}$

$= 2 \begin{vmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 2(1)(3)(-6)(1) =$

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