Because $AX = 0 \Rightarrow A^{-1}AX = A^{-1}0 = 0$

$\Rightarrow I_nX = 0 \Rightarrow X = 0$

On the other Hand consider the Matrix $A = \begin{pmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{pmatrix}$ which we have seen has no inverse.

~~There~~ are $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} := 0$

s.t

$AX = 0$ Because

$$\left(\begin{array}{ccc|c} 0 & 3 & -5 & 0 \\ 1 & 0 & 2 & 0 \\ -4 & -9 & 7 & 0 \end{array}\right) \xrightarrow{\text{Row Reduce}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

No Pivot in Row 3 so x_3 is free $x_3 = t \in \mathbb{R}$

$\Rightarrow 3x_2 = -5x_3 = -5t$

$\Rightarrow x_2 = -5/3\, t$

& $x_1 = -2x_3 = -2t$

So The Solns of $AX = 0$ are all vectors of the form $X = \begin{pmatrix} -2t \\ -5/3\, t \end{pmatrix}$

$= t \begin{pmatrix} -2 \\ -5/3 \\ 1 \end{pmatrix}$. The vectors X sati

$AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ for any Matrix A

are called the kernel of A.

In this example $\ker A := \{ X \in \mathbb{R}^3$

$AX =$

$= \text{Span}\left\{ \begin{pmatrix} -2 \\ -5/3 \\ 1 \end{pmatrix} \right\}$ i.e. is the line (1 dimensiona

in the direction of the vector

$(-2, -5/3, 1)$ (through the origin (when $t = 0$

Note: The dimension of the kernel

A + the rank A = 3

This is True for any $n \times n$ Matrix

$$\dim \ker A + \text{rank } A = n.$$

Recall the Matrix $A = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 2 & -3 & 4 & -3 \\ 3 & -5 & 5 & -4 \\ -1 & 1 & -3 & 2 \end{pmatrix}$

Row operations $\longrightarrow$ (see earlier lectures) $\begin{pmatrix} 1 & 0 & 5 & -3 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ so clearly

Rows 1 & 2 are linearly indep

$\Rightarrow$ rank A = & Solving $AX = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 5 & -3 & 0 \\ 0 & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \end{array}\right)$$

so only pivots in Rows 1 & 3 in x_1 & x_2 positions (non-free variables)

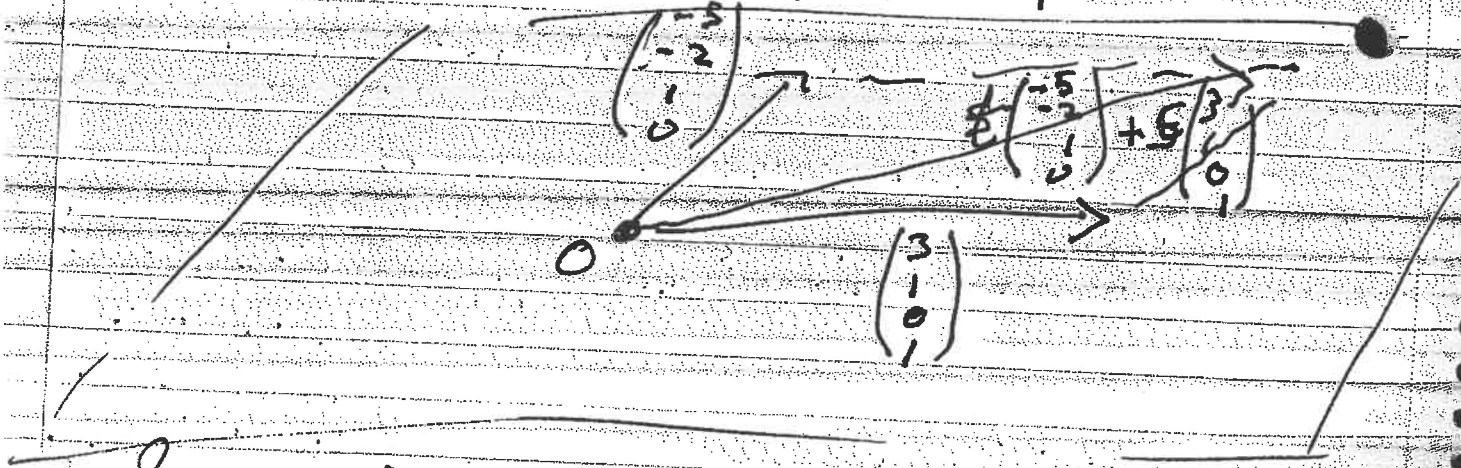
$\Rightarrow$ x_3 & x_4 are free & can assume any Real no.s t & $s \in \mathbb{R}$ as values respectively

Row 2 $\Rightarrow$ $x_2 = -2x_3 + x_4 = -2t + s$

& Row 1 $\Rightarrow$ $x_1 = -5x_3 + 3x_4 = -5t + 3s$

$$\Rightarrow X = \begin{pmatrix} -5t+3s \\ -2t+s \\ t \\ s \end{pmatrix} = t\begin{pmatrix}-5\\-2\\1\\0\end{pmatrix} + s\begin{pmatrix}3\\1\\0\\1\end{pmatrix}$$

i.e. a 2-dimensional Soln space to $AX = 0$



& again rank A + dim Ker A = n $\Rightarrow$

$2 + 2 = n$