§ Finding the inverse of a Matrix By Row Operations.

Recall that the inverse of an $n \times n$ Matrix A is another $n\times n$ Matrix denoted by A^{-1} s.t.

$$AA^{-1} = A^{-1}A = \underbrace{I_n}_{\text{(the identity matrix)}} := \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

We don't know A^{-1} so lets find its 3 Cols X_1, X_2, X_3 as follows.

$$A^{-1} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ X_1 & X_2 & X_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

So Since $AA^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow \quad AX_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; AX_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\& \quad AX_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 5 & -3 \\ -3 & 2 & -4 \end{pmatrix}$

Let $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = X_1$ Be the 1st Col of A^{-1}

we must have that $AX_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

ie To find $X_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ we Solve the system

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 1 \\ 2 & 5 & -3 & 0 \\ -3 & 2 & -4 & 0 \end{array}\right) \xrightarrow{\text{Put in}}$$

Reduced Echelon form to g

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right)$$

Now to find $X_2 = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ Solve

$AX_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ ie $\left(\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 2 & 5 & -3 & 1 \\ -3 & 2 & -4 & 0 \end{array}\right)$

$\xrightarrow{\text{Put in Reduced}}$ Echelon form to get

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & u \\ 0 & 1 & 0 & v \\ 0 & 0 & 1 & w \end{array}\right)$$

But we just did the same set of Row operations again! so:

Do All three at the same time ie

$$\left(\begin{array}{ccc|ccc} \boxed{1} & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -3 & 0 & 1 & 0 \\ -3 & 2 & -4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{Row reduce to}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A^{-1} \\ 0 & 0 & 1 & \end{array}\right)$$

ie.

$$\left(\begin{array}{ccc|ccc} \boxed{1} & 3 & -2 & 1 & 0 & 0 \\ 2 & 5 & -3 & 0 & 1 & 0 \\ -3 & 2 & -4 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 + 3R_1]{R_2 - 2R_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 \\ 0 & 11 & -10 & 3 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to -R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & -2 & 1 & 0 & 0 \\ 0 & \boxed{1} & -1 & 2 & -1 & 0 \\ 0 & 11 & -10 & 3 & 0 & 1 \end{array}\right) \xrightarrow[R_3 - 11R_2]{R_1 - 3R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & -5 & 3 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 \\ 0 & 0 & \boxed{1} & -19 & 11 & 1 \end{array}\right) \xrightarrow[R_2 + R_3]{R_1 - R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 14 & -8 & -1 \\ 0 & 1 & 0 & -17 & 10 & 1 \\ 0 & 0 & 1 & -19 & 11 & 1 \end{array}\right)$$
$$\underbrace{}_{A^{-1}}$$

So what can go wrong so that A^{-1} doesn't exist?

If A didn't have 3 Pivot (positions we wouldn't be able to reduce to $(I|A^{-1})$ e.g.

Let $A = \begin{pmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{pmatrix}$ Then

$$(A|I) = \left(\begin{array}{ccc|ccc} 0 & 3 & -5 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ -4 & -9 & 7 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2}$$

$$\left(\begin{array}{ccc|ccc} \boxed{1} & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ -4 & -9 & 7 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 + 4R_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & \boxed{3} & -5 & 1 & 0 & 0 \\ 0 & -9 & 15 & 0 & 4 & 1 \end{array}\right) \xrightarrow{R_3 + 3R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & -5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 4 & 1 \end{array}\right)$$

No 3rd Pivot & clearly inconsistent eqns.

Another way to Put the Problem with $\begin{pmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{pmatrix}$ Not

having an inverse is that we Reduced it to $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 0 & -9 & 15 \end{pmatrix}$

So that clearly Row 3 = -3 Row 2 ie not linearly independant

OR in otherwords $\longrightarrow$

$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{pmatrix}$ ie we were able to find a non trivial linear combination

$$c_1 R_1 + c_2 R_2 + c_3 R_3 = 0 \; 0 \; 0$$

ie the Rows are not linearly indep.

The number of linearly independent Rows of a Matrix doesn't change on Performing Row operations But it makes it easier to spot linearly independent rows. e.g.

$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 0 \end{pmatrix}$ has clearly 2 linea[rly]

indep rows (as 1 0 2 can't [be]

~~a multiple of 0 3 5)~~ so th[e]

original matrix $\begin{pmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{pmatrix}$

Defn: Let A be an $n \times n$ Ma[trix]

The Rank of A denoted rank A

:= the no. of linearly indep rows

of A

(Aside: it can be shown that rank A

is also equal to the no of l.i column[s]

of A)

Thm: A^{-1} exists i.e. A is an

invertible $n \times n$ Matrix $\Longleftrightarrow$ rank $A = n$

Note If A^{-1} exists and $X = \begin{pmatrix} \cdot \\ \vdots \\ \cdot \end{pmatrix}$

Then $AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$

$\Rightarrow X = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} := 0$

Because $AX = 0 \Rightarrow A^{-1}AX = A^{-1}0 = 0$

$\Rightarrow I_n X = 0 \Rightarrow X = 0$

On the other Hand consider the

Matrix $A = \begin{pmatrix} 0 & 3 & -5 \\ 1 & 0 & 2 \\ -4 & -9 & 7 \end{pmatrix}$ which

we have seen has no inverse.

~~There~~ are $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} := 0$

s.t

$AX = 0$ Because

$$\left(\begin{array}{ccc|c} 0 & 3 & -5 & 0 \\ 1 & 0 & 2 & 0 \\ -4 & -9 & 7 & 0 \end{array}\right) \xrightarrow{\text{Row Reduce}}$$