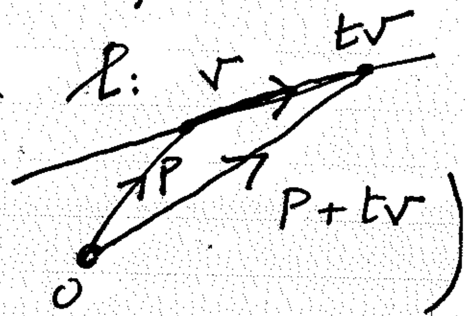
OBSERVE: (i) A Linear Transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ sends lines to lines Because given a line ℓ in $\mathbb{R}^n$ (in parametric form)

$\ell : P + tv,\ t \in \mathbb{R}$ (ie ℓ:



)

$L(\ell) = L(P) + L(tv)$

$= L(P) + t L(v).$

This is the Parametric form of The line (in $\mathbb{R}^m$) through the Point $L(P)$ in the direction $L(v)$

(if $L(v) = 0$ $(0 = (0,0,\ldots,0) \in \mathbb{R}^m)$

(ii) L maps the zero vector in $\mathbb{R}^n$ to the zero vector in $\mathbb{R}^m$.

Because $L(0+0) = L(0)$

$$\Rightarrow \quad L(0) + L(0) = L(0)$$

$$\Rightarrow \quad L(0) = 0$$

Alternatively using the matrix $A \longleftrightarrow L$

$L(0)$ is obtained as:

$$A \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^m$$

Examples of Linear Transformations.

Ex: Fix a vector $n = (n_1, n_2, n_3) \in \mathbb{R}^3$

Define $L : \mathbb{R}^3 \to \mathbb{R}^3$ as follows:

$$L : v \longrightarrow n \times v$$ (L is linear

as $n \times (v + w) = n \times v + n \times v$

& $n \times (kv) = k(n \times v)$ $\quad k \in \mathbb{R}$.

To Find the Matrix A_L for L (w.r.t the Standard Basis $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$

we Find $L(e_1)$, $L(e_2)$ & $L(e_3)$ & then

$$A_L = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ L(e_1) & L(e_2) & L(e_3) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$L(e_1) = n \times e_1 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= e_1\left((n_2)(0) - (0)(n_3)\right) - e_2\left((n_1)(0) - (1)(n_3)\right) + e_3\left(n_1)(0) - 1(n_2)\right)$$

$$= e_1(0) + n_3 e_2 - n_2 e_3 = (0, n_3, -n_2)$$

So 1st Col of A_L is $\begin{pmatrix} 0 \\ n_3 \\ -n_2 \end{pmatrix}$

Next $L(e_2) = n \times e_2 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 0 & 1 & 0 \end{vmatrix} =$

$$e_1\big((n_2)(0) - (1)(n_3)\big) - e_2\big((n_1)(0) - (0)(n_3)\big) + e_3\big((n_1)(1) - (0)(n_2)\big)$$

$$= -n_3 e_1 - 0 e_2 + n_1 e_3 = (-n_3, 0, n_1$$

So The second col of A_L is

$$\begin{pmatrix} 0 & -n_3 \\ n_3 & 0 \\ -n_2 & n_1 \end{pmatrix}$$

& Finally

$L(e_3) = n \times e_3 = \begin{vmatrix} e_1 & e_2 & e_3 \\ n_1 & n_2 & n_3 \\ 0 & 0 & 1 \end{vmatrix} =$

$$e_1\big((n_2)(1) - (0)(n_3)\big) - e_2\big((n_1)(1) - (0)(n_3)\big) + e_3\big((n_1)(0) - (0)(n_2)\big)$$

$= n_2 e_1 - n_1 e_2 + 0 e_3$ So 3rd Col of A_L is

$$\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix}$$

(This is a Skew Symmetrix MATRIX ie $A_L^T = -A_L$)

$$A_L^T = \begin{pmatrix} 0 & n_3 & -n_2 \\ -n_3 & 0 & n_1 \\ n_2 & -n_1 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{pmatrix} = -A_L$$

Recall from last year the Matrix A_L for a Rotation L in $\mathbb{R}^2$ about the origin ↺ by an angle θ.

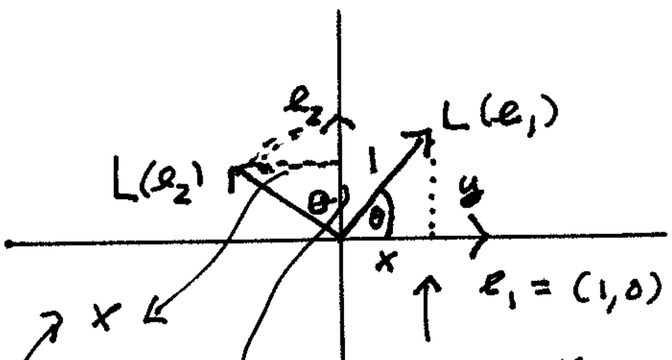
We need to find $L(e_1) = L((1,0))$

$L(e_2) = L((0,1))$

&

& then

$$A_L = \begin{pmatrix} \uparrow & \uparrow \\ L(e_1) & L(e_2) \\ \downarrow & \downarrow \end{pmatrix}$$



$L(e_1) = (x, y) = (\cos\theta, \sin\theta)$

$L(e_2) = (-\sin\theta, \cos\theta)$

$\cos\theta = \frac{x}{1} = x$

$\sin\theta = y/1 = y$

$x = -\sin\theta$ &

$y = \cos\theta$

$\Rightarrow L(e_2) = (-\sin\theta, \cos\theta)$

So $$A_L = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Ex: Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ Be rotation about the e_3 axis ↺ By an angle θ. Then in the $e_1 - e_2$ plane we saw Last year that.

$$L(e_1) = \cos\theta\, e_1 + \sin\theta\, e_2 + 0 e_3$$

& so the 1st Col of A_L is $\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$

& $L(e_2) = -\sin\theta\, e_1 + \cos\theta\, e_2 + 0 e_3$

So the 2nd Col of A_L is $\begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$

& Finally $L(e_3) = e_3$

$= 0 e_1 + 0 e_2 + 1 e_3$ $\therefore$ 3rd Col of A_L is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ie

$$A_L = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly the Matrix for a rotation about the e_1 axis is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$

Ex: Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ Be Projection of $\mathbb{R}^3$ onto the $x_1 - x_2$ Plane

ie $L((x_1, x_2, x_3)) = (x_1, x_2)$

so $L((1,0,0)) = (1,0)$

$L((0,1,0)) = (0,1)$

& $L((0,0,1)) = (0,0)$

So $A_L = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ L(e_1) & L(e_2) & L(e_3) \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

§ Revision of Matrix Multiplication:

Recall that Matrix Multiplication was so defined so that if

$L_1 \leftrightarrow A$ & $L_2 \leftrightarrow B$

Then $L_1 \circ L_2 \leftrightarrow AB$ ($\neq BA$ usually)

So if $v = (x_1, x_2, \ldots, x_n)$ to find

$L_1 \circ L_2 (v) := L_1(L_2(v))$ we place

($L_2 : \mathbb{R}^n \to \mathbb{R}^m$ & $L_1 : \mathbb{R}^m \to \mathbb{R}^p$)

$v = (x_1, x_2, \ldots x_n)$ Behind the matrix AB as a Column & Multiply

$$\underset{p\times \cancel{m}\, \cancel{m}\times n \quad n\times 1}{AB \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}} := \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}$$

$= p \times \cancel{n}\ n \times 1$

$= p \times 1$

& $L_1 \circ L_2 (v) = (y_1, \ldots, y_p)$

Recall : AB only makes sense if The number of entries in a Row of A = The number of entries in a Column of B ie if A is $p \times m$ & B is $m \times n$

$\therefore AB$ is $p \times \underbrace{\cancel{m}\ \cancel{m}}_{\text{equal}} \times n = p \times n$

If $R = r_1 r_2 \ldots r_m$ is a Row of A

& $C = \begin{matrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{matrix}$ is a Col of B

Then $RC := r_1 c_1 + r_2 c_2 + \cdots + r_m c_m$

& we obtain the entries of AB by multiplying all the rows of A by all the columns of B & placing the answer in the corresponding position of AB.

e.g. ith row of A multiplied by the jth col of B is placed in the ith row & jth col of AB.

Ex:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$

$$2\times \underbrace{3 \qquad\qquad 2}_{\text{no equal}} \times 2$$

So AB doesn't make sense But BA

$$2\times \underbrace{\cancel{2} \; \cancel{2}}_{\text{equal}} \times 3$$

So BA makes sense & is a 2×3 matrix

$$BA = \begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} (2\ 3)\begin{pmatrix}1\\3\end{pmatrix} & (2\ 3)\begin{pmatrix}2\\-1\end{pmatrix} & (2\ 3)\begin{pmatrix}1\\2\end{pmatrix} \\ (1\ 1)\begin{pmatrix}1\\3\end{pmatrix} & (1\ 1)\begin{pmatrix}2\\-1\end{pmatrix} & (1\ 1)\begin{pmatrix}1\\2\end{pmatrix} \end{pmatrix} =$$

$$\begin{pmatrix} 2+9 & 4-3 & 2+6 \\ 1+3 & 2-1 & 1+2 \end{pmatrix} = \begin{pmatrix} 11 & 1 & 8 \\ 4 & 1 & 3 \end{pmatrix}$$