MA 203 LINEAR ALGEBRA:

COURSE TEXT: LINEAR ALGEBRA & its APPLICATIONS.
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Assessment: 70% EXAM

30% CA (3 class TESTS based on TUTORIAL SHEETS.)

TUTORIALS: WED 10 & THURS 1.

Course Outline: We will cover chapters 1 to 3 (& some of chapter 4) of the course text.

§ Introduction: Geometric Background ie vectors and the dot product.

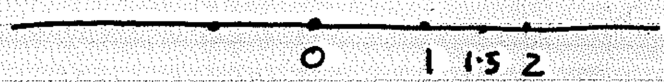
Recall $\mathbb{R}$ denotes the set of all real numbers & $\mathbb{N}$ the set of natural numbers, ie $\mathbb{N} = \{1, 2, 3, \ldots\}$

Let $n \in \mathbb{N}$

$$\mathbb{R}^n := \{ (x_1, x_2, \ldots, x_n) \mid x_i \in \mathbb{R} \}$$

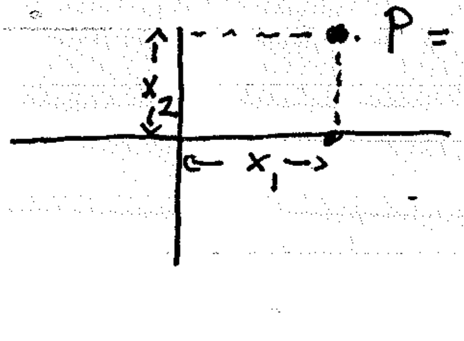
An element $v = (x_1, \ldots, x_n) \in \mathbb{R}^n$ is called a vector.

PICTURES $\mathbb{R}$ $(n = 1)$



$n = 2$ $\mathbb{R}^2 = \{ (x_1, x_2) \mid x_1, x_2 \in \mathbb{R} \}$

PICTURE: (a) As the Cartesian plane



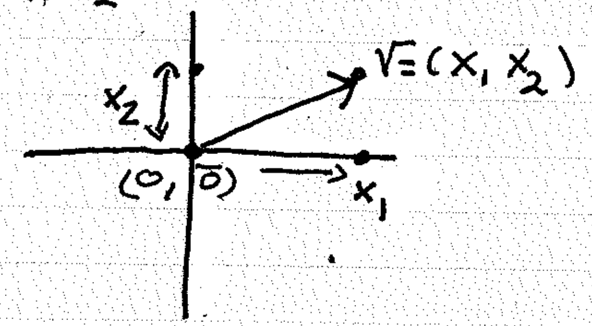
$P = P(x_1, x_2) = (x_1, x_2)$

(Also called 2-dim Euclidean space)

OR

Picture (b) as a 2 dimensional vector $v = (x_1, x_2)$

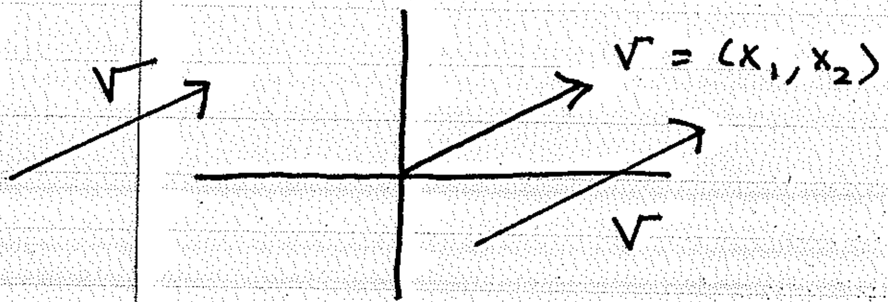
ie the directed line segment from (0 0) to (x_1, x_2) in the Cartesian Plane



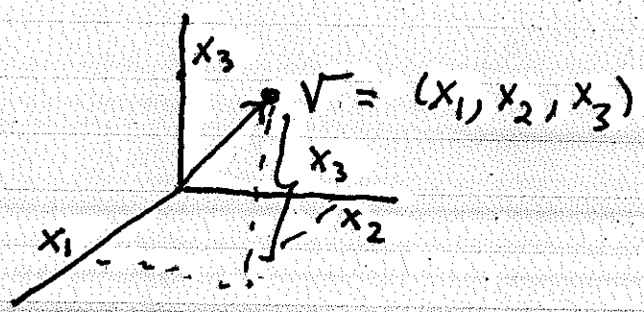
(so its a quantity with direction & magnitude or length)

OR

Picture (c) Base the vector Picture at any point in the Plane



$n = 3$



Any element $v = (x_1, x_2, .., x_n) \in \mathbb{R}^n$ is called an n-dimensional vector

OPERATIONS on VECTORS.

(i) Scalar Multiplication (or Scaling)

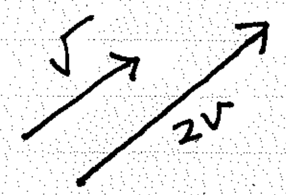
If $v = (x_1, x_2, .., x_n) \in \mathbb{R}^n$ & $r \in \mathbb{R}$

$rv := (rx_1, rx_2, \cdots, rx_n) \in \mathbb{R}^n$ is a vector in the same direction as v (if $r > 0$) But r times as long.

(If $r < 0$, it is in the opposite direction to v But r times as long)

Ex $\quad v = (1,2,3) \quad r = 2.$

$\quad 2v = (2,4,6)$



(ii) Addition of vectors:

Let $v = (v_1, v_2, \ldots, v_n)$ & $u = (u_1, u_2, \ldots, u_n)$

Then $v + u := (v_1 + u_1, v_2 + u_2, \ldots, v_n + u_n)$

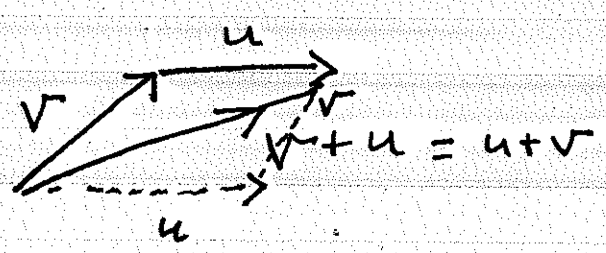
e.g. Note $v + u = u + v$. E.g

$v = (1,2) \quad u = (2,1)$

$v + u = (1+2, 2+1) = (3,3)$

$u + v = (2+1, 1+2) = (3,3)$

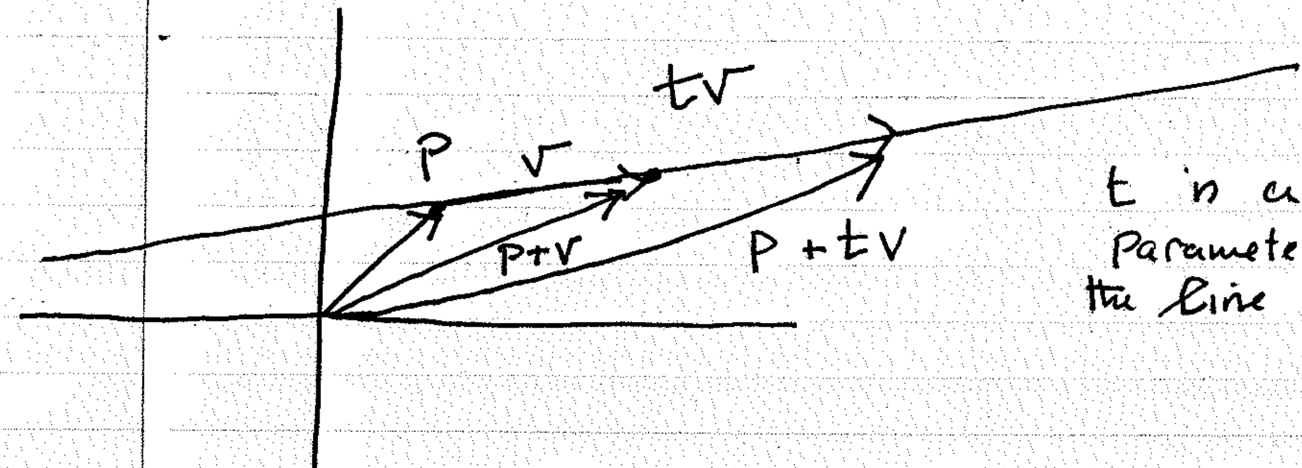
Geometric Picture: Place Place u at the tip of v & $v+u$ is the vector from the Base of v to the tip of u



Application: The Parametric equation of a line in $\mathbb{R}^n$

We will draw it in the case $n = 2$

To describe a line ℓ we need to specify the direction of the line $v \neq 0$ and a point P on the line



t in a parameter for the line

So $\ell = \{ P + tv \mid t \in \mathbb{R} \}$.

Ex Find the line (ie the parametric eqn) in $\mathbb{R}^3$ in the direction $v = (1,2,3)$ passing through the point $P = (2,4,6)$

$$\ell = \{ (2,4,6) + t(1,2,3) \mid t \in \mathbb{R} \}$$
$$= \{ (2,4,6) + (t, 2t, 3t) \mid t \in \mathbb{R} \}$$
$$= \{ (2+t, 4+2t, 6+3t) \mid t \in \mathbb{R} \}$$

ie ℓ consists of the points (x_1, x_2, x_3) in $\mathbb{R}^3$ where

$$x_1 = 2+t \ , \ x_2 = 4+2t \ \& \ x_3 = 6+3t$$

$$\Rightarrow \quad t = \frac{x_1 - 2}{1} = \frac{x_2 - 4}{2} = \frac{x_3 - 6}{3}$$

So In general If $P = (P_1, P_2, P_3)$ & $v = (v_1, v_2, v_3)$

The Eqn is $(t =) \quad \dfrac{x_1 - P_1}{v_1} = \dfrac{x_2 - P_2}{v_2} = \dfrac{x_3 - P_3}{v_3}$

Question: ~~To~~ Be answered later By Solving Simultaneous linear equations

let $\ell_1 : \dfrac{x_1 - 3}{4} = \dfrac{x_2 - 1}{2} = \dfrac{x_3 + 1}{3}$

& $\ell_2 : x_1 - 1 = \dfrac{x_2 - 2}{4} = \dfrac{x_3 - 1}{2}$

Do ℓ_1 & ℓ_2 intersect & if so find their intersection.

In Parametric form we have

$\ell_1 : (3, 1, -1) + t(4, 2, 3)$ &

$\ell_2 : (1, 2, 1) + s(1, 4, 2)$ so where

They meet the x_1, x_2 & x_3 coordinates must be the same ie

$$\begin{aligned} 3 + 4t &= 1 + s \\ 1 + 2t &= 2 + 4s \\ -1 + 3t &= 1 + 2s \end{aligned}$$ &

OR

$$\begin{aligned} 4t - s &= -2 \\ 2t - 4s &= 1 \\ 3t - 2s &= 2 \end{aligned}$$

We don't expect any Solns as: Geometrically in general 2 lines in $\mathbb{R}^3$ won't meet & Too many eqns ie 3 eqns in 2 unknowns

§ THE DOT PRODUCT: (Encodes length and Angle Between vectors)

Defn: Let $u = (u_1, u_2, \dots, u_n)$ & $v = (v_1, \dots, v_n)$

Then the dot or scalar product of u & v is the real number denoted by $u \cdot v$ & defined by

$$u \cdot v = u_1 v_1 + u_2 v_2 + \dots + u_n v_n := \sum_{i=1}^{n} u_i v_i$$

Ex $u = (1, 2, 1)$ $v = (-1, 2, 2)$

$u \cdot v = (1)(-1) + (2)(2) + (1)(2) = 5$

& $v \cdot u = 5$

Ex: $u = (2, 3)$ $v = (-3, 2)$ then

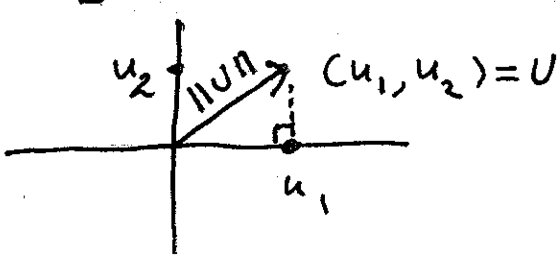
$u \cdot v = -6 + 6 = 0$

& $u \cdot u = 2^2 + 3^2$ = the distance squared from $u = (2, 3)$ to $(0, 0)$ ie the length $\|u\|$ of the vector u squared.

In general by Pythagorous we have if $u = (u_1, u_2) \in \mathbb{R}^2$ then

$\|u\|^2 = u_1 u_1 + u_2 u_2 = u_1^2 + u_2^2 \implies$

$\|u\| = \sqrt{u_1^2 + u_2^2}$



Defn: Let $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, the norm or length of v denoted $\|v\|$ is defined to be $\|v\| = \sqrt{v \cdot v} = \sqrt{v_1^2 + \cdots + v_n^2}$.

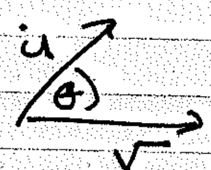
Properties of the Dot Product

(i) $u \cdot v = v \cdot u \quad \forall\, u, v \in \mathbb{R}^n$

(ii) $u \cdot (v + w) = u \cdot v + u \cdot w \quad \forall\, u, v, w \in \mathbb{R}^n$

(iii) $(ru) \cdot v = u \cdot (rv) = r(u \cdot v) \quad \forall\, u, v \in \mathbb{R}^n,\ r \in \mathbb{R}$

(iv) In $\mathbb{R}^2$ we will prove that
$u \cdot v = \|u\| \|v\| \cos\theta$ where θ in the angle between u & v



As in higher dimensions any 2 vectors define a 2-dimensional plane we also have the same notion of angle θ

Corollary: $u \cdot v = 0 \iff \cos\theta = 0$
$\iff$ ie $\theta = \frac{\pi}{2}$

So u & v are perpendicular or orthogonal.

Defn: Given a vector $n = (n_1, n_2, n_3, \ldots n_n)$ in $\mathbb{R}^n$ the hyperplane through