Recall: If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then If $|A| \neq 0$

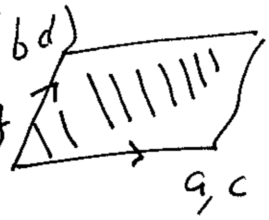
$$A^{-1} = \frac{1}{|A|} A^{*} = \frac{1}{|A|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{(ad-bc)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

We will now Prove (not examiniable) that

$|A| = \pm$ the area of the Parallelogram

with sides the vectors (a,c) & (b,d)

ie $|A| = \pm$ Area of

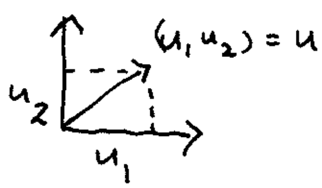


Firstly: We introduce the dot Product of 2 vectors $u = (u_1, u_2)$ & $v = (v_1, v_2)$

Then the dot (or scalar) product of u & v is denoted By $u \cdot v$ & is defined By

$$u \cdot v = u_1 v_1 + u_2 v_2 \qquad \left(= (u_1, u_2)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \text{ Row x Col} \right)$$

Note

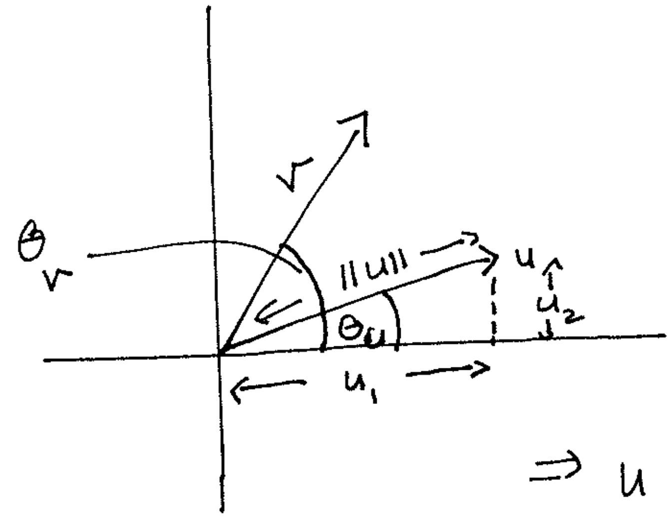


then $u \cdot u = u_1^2 + u_2^2 = (\|u\|)^2$
where $\|u\|$ denotes the length of u (By Pythagarous)

ie (i) $\|u\| = \sqrt{u \cdot u}$. We now show:

(ii) $u \cdot v = \|u\|\|v\| \cos\theta$ where θ is the angle Between u & v

Proof of (ii)



$$\cos\theta_u = \frac{u_1}{\|u\|} \quad \& \quad \sin\theta_u = \frac{u_2}{\|u\|}$$

$$\Rightarrow u = \|u\| (\cos\theta_u, \sin\theta_u)$$

& Similarly $\quad v = \|v\| (\cos\theta_v, \sin\theta_v)$

& from the defn of $u \cdot v$ we get that

$$u \cdot v = \|u\| \|v\| (\underbrace{\cos\theta_u \cos\theta_v + \sin\theta_u \sin\theta_v}_{= \cos(\theta_v - \theta_3) \text{ (log tables)}})$$

& $\theta_v - \theta_u = \theta$ the angle between u & v.

<u>Corollary</u>: $u, v \in \mathbb{R}^2$ are perpendicular OR orthogonal if and only if (written $\Longleftrightarrow$)

$$u \cdot v = \|u\| \|v\| \cos\tfrac{\pi}{2} = 0.$$ We write $u \perp v$

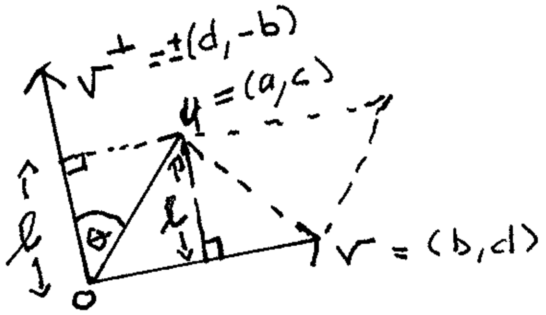
Note If $v = (v_1, v_2)$ then $v^\perp = \pm(v_2, -v_1)$ is perpendicular to v because

$$v \cdot v^\perp = v_1 v_2 - v_1 v_2 = 0$$ and

$$\|v\| = \|v^\perp\| = \sqrt{v_1^2 + v_2^2}.$$

Propⁿ: Let $u = (a,c)$ & $v = (b,d)$, then the area of the Parallelogram with sides u & v is $\pm\,(ad - bc)$.

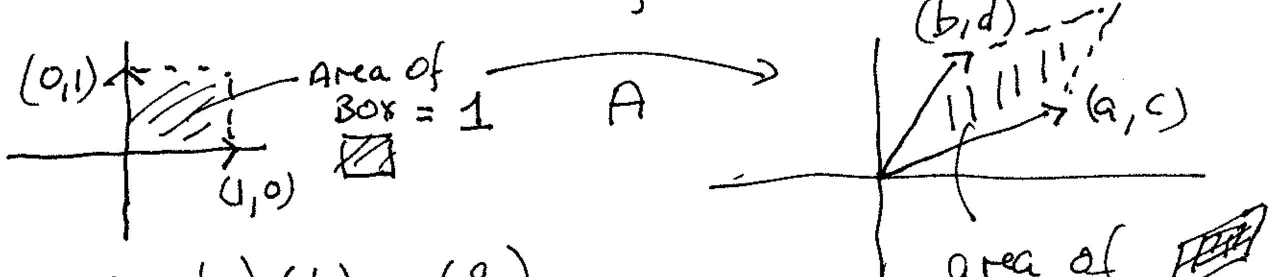
Proof:



The Area of the Parallelogram = 2 Area of the $\triangle$ with Sides u & v

$$= 2 \cdot \tfrac{1}{2} \|v\| \ell$$
$$= \|v^{\perp}\| \ell$$
$$= \|v^{\perp}\| \, \|u\| \cos\theta$$
$$= v^{\perp} \cdot u$$
$$= \pm(d,-b)\cdot(a,c) = \pm\,(ad - bc)$$

NOTE: THIS implies the for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the determinant $|A| = ad - bc$ gives the change of area under the transformation described By A.



area of ▱ $= \pm\,(ad - bc)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$$

THIS FACT GIVES A Geometric interpretation of the formula $|AB| = |A||B|$:

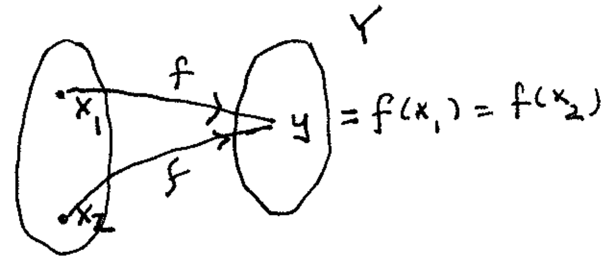
If we first apply B & then apply A area will change by a factor of $|A||B|$. But this is just the composition of the two linear map $A \circ B$ which has matrix AB so the change of area equals $|AB|$

so $|AB| = |A||B|$

Also if A^{-1} exists then clearly $|A^{-1}| = \frac{1}{|A|}$ as if A distorts area by a factor $|A|$

A^{-1} distorts area " " " $\frac{1}{|A|}$

Also if a fn $f : X \to Y$ between 2 sets X & Y sends 2 different values $x_1 \neq x_2$ to the same image ie $f(x_1) = f(x_2)$



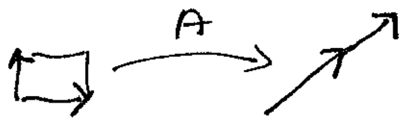
Then f^{-1} can't exist as ~~to~~ undo f we need f^{-1} to send $y = f(x_1) = f(x_2)$ to both x_1 & x_2. But a fn can't send one pt to 2.

If $A : \mathbb{R}^2 \to \mathbb{R}^2$ is such that $|A| = 0$

then the image of 

has area $= 0$ as the change of area $= |A|$ so its in one dimensional

ie $A\binom{1}{0} = rA\binom{0}{1}$ for some $r \in \mathbb{R}$



$= A\left(r\binom{0}{1}\right)$

ie $(1,0)$ & $r(0,1)$ get mapped by A to the same vector

so A^{-1} can't exist.