A row $(e\ f)$ Multiplied By a Column $\begin{pmatrix} g \\ h \end{pmatrix}$
$(\times)$

$$:= eg + fh \quad (\in \mathbb{R})$$

Now to Find the Matrix AB multiply the 2 Rows of A (R_1 & R_2) By the 2 Columns of B (C_1 & C_2) & Place the numbers obtained in the Corresponding position of AB ie

$$AB := \begin{pmatrix} R_1 \times C_1 & R_1 \times C_2 \\ R_2 \times C_1 & R_2 \times C_2 \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $B = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$

$$AB = \begin{pmatrix} (1\ 2)\begin{pmatrix} 2 \\ 1 \end{pmatrix} & (1\ 2)\begin{pmatrix} -1 \\ 3 \end{pmatrix} \\ (3\ 4)\begin{pmatrix} 2 \\ 1 \end{pmatrix} & (3\ 4)\begin{pmatrix} -1 \\ 3 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 5 \\ 10 & 9 \end{pmatrix}$$

Recall $A = \begin{pmatrix} 2 & -3 \\ 0 & 5 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}$

$$AB = \begin{pmatrix} 5 & -11 \\ -5 & 25 \end{pmatrix} \quad BA = \begin{pmatrix} 1 & 2 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 7 \\ -2 & 28 \end{pmatrix}$$

$$AB \neq BA$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad AI = IA = A$$

$\forall$ matrices A.

Ex: $L((x,y)) = (ax+by,\ cx+dy)$

$$L \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

The image of $v = (x,y)$ under L is obtained as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

Rewrite as $(ax+by, cx+dy)$

Ex: Find the image of the vector $(2,1)$ under the Linear Transformation

$$L((x,y)) = (2x - 3y,\ 5y)$$

$$\updownarrow$$

$$\begin{pmatrix} 2 & -3 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

So $(1, 5)$ is the image of $(2,1)$

§ Systems of Linear Equations:

Ex: Solve

$$\begin{aligned} 2x + y &= 3 \\ x + 4y &= 2 \end{aligned} \qquad (+)$$

Rewrite in terms of matrices:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

(+) Becomes: $AX = b$

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

How to Find A^{-1}:

$$A^{-1}A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

(note: $\begin{pmatrix} -c & a \end{pmatrix}\begin{pmatrix} a \\ c \end{pmatrix} = -ac + ac = 0$

$$\boxed{A^{*} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}} \qquad \boxed{|A| := ad - bc}$$

(adjoint of A)

The determinant of A.

So $\boxed{A^{-1} := \frac{1}{|A|} A^{*}}$

$\underline{\underline{Ex}}$: $A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$

$$|A| = (2)(4) - (1)(1) = 7$$

$$A^{-1} = \frac{1}{|A|} A^{*} = \frac{1}{7}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$$

(check: $\underbrace{\frac{1}{7}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}}_{A^{-1}} \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So To Solve: $\begin{aligned} 2x + y &= 3 \\ x + 4y &= 2 \end{aligned}$ $\quad b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Rewrite:

$X = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad AX = b \quad A = \begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix}$

$$A^{-1}AX = A^{-1}b$$

$$IX = A^{-1}b$$

So $\quad X = A^{-1} b$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 10 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{10}{7} \\ \frac{1}{7} \end{pmatrix}$$

$$x = \frac{10}{7} \qquad y = \frac{1}{7}$$

Check:

$$2\left(\frac{10}{7}\right) + \frac{1}{7} = \frac{21}{7} = 3 \checkmark$$

$$\frac{10}{7} + \frac{4}{7} = \frac{14}{7} = 2 \checkmark$$

FACT: A has an inverse if and only if $|A| \neq 0$.