Some Sample Questions on HW4:

Ex: Let $A = \begin{pmatrix} 0 & 4 \\ x & 0 \end{pmatrix}$. Find the value of x s.t. $\lambda = 2$ is an e-value of A.

Recall: λ is an e-value of $A \iff |A - \lambda I| = 0$

So need $|A - 2I| = \left| \begin{pmatrix} -2 & 4 \\ x & -2 \end{pmatrix} \right| = 0$

$\Rightarrow \quad 4 - 4x = 0 \quad \Rightarrow \quad x = 1$

Ex: Find the entry in the first Row & first Col. of A^{10} where $A = \begin{pmatrix} -3 & 5 \\ -2 & 4 \end{pmatrix}$

Warning: THE Entries of A^{10} are NOT! those of A Raised to the power 10. This is only True if A is diagonal.

Step I: Find the e-values of A ie Solve $|A - \lambda I| = 0$ (The characteristic eqn of A)

ie $\left| \begin{pmatrix} -3-\lambda & 5 \\ -2 & 4-\lambda \end{pmatrix} \right| = 0$

$\Rightarrow \quad -12 - \lambda + \lambda^2 + 10 = 0 \Rightarrow \lambda^2 - \lambda - 2 = 0$

$\Rightarrow \quad (\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda_1 = -1, \ \lambda_2 = 2$

Step II: Find corresponding e-vectors

ie Solve $(A - \lambda I)v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $v = \begin{pmatrix} x \\ y \end{pmatrix}$

For $\lambda = -1$ have $\begin{pmatrix} -2 & 5 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$(A - (-1)I)$
$\parallel$
$(A + I)$

$\Rightarrow \quad -2x + 5y = 0 \quad \Rightarrow 2x = 5y$

$(\ \& \ -2x + 5y = 0 \, ! \ )$

e.g. $v_1 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$

$(A - 2I)$

& For $\lambda = 2$ have $\begin{pmatrix} -5 & 5 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad 5x = 5y \quad \Rightarrow x = y$ e.g.

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

<u>Now</u> Let $E = \begin{pmatrix} 5 & 1 \\ 2 & 1 \end{pmatrix}$

$D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}$

$AE = ED$
$\Rightarrow A = EDE^{-1}$

Then $A^{10} = \underbrace{EDE^{-1}EDE^{-1}E \ldots E^{-1}EDE^{-1}}_{10 \text{ times}}$

$= ED^{10}E^{-1}$

$= \frac{1}{3}\begin{pmatrix} 5 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{10} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2 & 5 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 5 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -2^{11} & (5)2^{10} \end{pmatrix}$

So (1,1) entry of A^{10} $= \frac{1}{3}(5 \ \ 1)\begin{pmatrix} 1 \\ -2^{11} \end{pmatrix} = \frac{1}{3}(5 - 2^{11})$

THE CAYLEY-HAMILTON Theorem: Let A Be a 2×2 matrix with characteristic equation $|A - \lambda I| = 0 = \lambda^2 + b\lambda + c$.
Then A "satisfies" its characteristic eqn in the following sense:

$$A^2 + bA + cI = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$