§ THE INVERSE OF A 3×3 Matrix (if it exists)

Problem: Given a 3×3 Matrix $A = (a_{ij})$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Find a 3×3 Matrix denoted By A^{-1} (if possible) s.t. $AA^{-1} = A^{-1}A = I$ where $I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. As in the 2×2 case we will define the determinant $|A|$ of A & A^{-1} will exist $\iff |A| \neq 0$. To define $|A|$ we first define

Defn: Let A Be a 3×3 matrix $A = (a_{ij})$

d_{ij} := the determinant of the 2×2 matrix obtained from A By deleting the i-th row & j-th Col of A.

Ex: $d_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{31}a_{12}$

Defn: $|A| := a_{11}d_{11} - a_{12}d_{12} + a_{13}d_{13}$ (& as in 2×2 case it equals $\pm$ Volume of the Box with Sided the columns of A.

Thm: If $|A| \neq 0$; $A^{-1} = \frac{1}{|A|} \left((-1)^{i+j} d_{ij} \right)^T$

where for a 3×3 matrix M
M^T is the transpose of M, ie its columns are the rows of M (Maintaining ORder)

Ex: Q.2 (b) in EXAM: Find A^{-1} where

$$A = \begin{pmatrix} 4 & 6 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 4 \end{pmatrix}$$

$$d_{11} = \begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} = 4 \qquad d_{12} = \begin{vmatrix} 2 & 6 \\ 6 & 4 \end{vmatrix} = -28 \qquad d_{13} = \begin{vmatrix} 2 & 4 \\ 6 & 2 \end{vmatrix} = -$$

$$d_{21} = \begin{vmatrix} 6 & 2 \\ 2 & 4 \end{vmatrix} = +20 \qquad d_{22} = \begin{vmatrix} 4 & 2 \\ 6 & 4 \end{vmatrix} = 4 \qquad d_{23} = \begin{vmatrix} 4 & 6 \\ 6 & 2 \end{vmatrix} = -2$$

$$d_{31} = \begin{vmatrix} 6 & 2 \\ 4 & 6 \end{vmatrix} = 28 \qquad d_{32} = \begin{vmatrix} 4 & 2 \\ 2 & 6 \end{vmatrix} = 20 \qquad d_{33} = \begin{vmatrix} 4 & 6 \\ 2 & 4 \end{vmatrix} = 4$$

$$\therefore |A| = a_{11}d_{11} - a_{12}d_{12} + a_{13}d_{13} = 4(4) - 6(-28) + 2(-20)$$
$$= 144$$

so $$A^{-1} = \frac{1}{144} \begin{pmatrix} 4 & -(-28) & -20 \\ -(+20) & 4 & -(-28) \\ 28 & -(20) & 4 \end{pmatrix}^T =$$

$$= \frac{1}{144} \begin{pmatrix} 4 & -20 & 28 \\ 28 & 4 & -20 \\ -20 & 28 & 4 \end{pmatrix}$$

So the entry in 3rd row & 3rd col of A^{-1} =

$$\frac{4}{144} = \frac{1}{36}$$

CHECK THAT $AA^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$