More Examples & Applications of Eigenvalues & Eigenvectors.

Ex: Prove that $v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ & $v_2 = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$ are eigenvectors of $A = \begin{pmatrix} 6 & 5 \\ -3 & -2 \end{pmatrix}$

and hence find A^{100}.

<u>Recall</u>: $v \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is an eigenvector of A with eigenvalue λ if and only if

$$Av = \lambda v \qquad \lambda \in \mathbb{R}.$$

$$Av_1 = \begin{pmatrix} 6 & 5 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1\begin{pmatrix} -1 \\ 1 \end{pmatrix} = 1v_1$$

So Yes v_1 is an eigenvector with eigenvalue $\lambda = 1$

& $$Av_2 = \begin{pmatrix} 6 & 5 \\ -3 & -2 \end{pmatrix}\begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -15 \\ 9 \end{pmatrix} = 3\begin{pmatrix} -5 \\ 3 \end{pmatrix} = 3v_2$$

So Yes v_2 is an eigenvector of A with eigenvalue $\lambda = 3$

Let $D := \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ & $E := \begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix}$

then $$AE = ED \Rightarrow A = EDE^{-1}$$

$$\therefore \quad A^{100} = ED\cancel{E^{-1}E}D\cancel{E^{-1}E}D\cancel{E^{-1}E} \cdots \cancel{E^{-1}E}DE^{-1}$$

$$= ED^{100}E^{-1}$$

$$= -\frac{1}{2}\begin{pmatrix} -1 & -5 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 3^{100} \end{pmatrix}\begin{pmatrix} 3 & 5 \\ -1 & -1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} -3-5(3^{100}) & -5-5(3^{100}) \\ 3-3^{101} & 5-3^{101} \end{pmatrix}$$

Ex: Calculating A^n is a very common problem e.g. in the theory of Markov Processes:

In a certain country there are 2 political parties F & G and everybody supports one or other. Polls suggest that every month $\frac{1}{4}$ of voters switch from $F \to G$. & $\frac{1}{3}$ of voters from $G \to F$

Let f_n & g_n denote the proportions of voters supporting F & G respectively in month n

Find the long term proportions of F & G voters $(n \to \infty)$

$$f_n = \tfrac{3}{4} f_{n-1} + \tfrac{1}{3} g_{n-1}$$

$$g_n = \tfrac{1}{4} f_{n-1} + \tfrac{2}{3} g_{n-1}$$

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} = \underbrace{\begin{pmatrix} 3/4 & 1/3 \\ 1/4 & 2/3 \end{pmatrix}}\begin{pmatrix} f_{n-1} \\ g_{n-1} \end{pmatrix}$$

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} = T \begin{pmatrix} f_{n-1} \\ g_{n-1} \end{pmatrix}$$

(T is called the transition Matrix FOR the Markov Process,

$$= T^2 \begin{pmatrix} f_{n-2} \\ g_{n-2} \end{pmatrix}$$

$$\vdots$$

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} = T^n \begin{pmatrix} f_0 \\ g_0 \end{pmatrix}$$

So need to know T^n when $n \longrightarrow \infty$.

To Find T^n Need to find the eigen value λ_1, λ_2 of T

$\longrightarrow$ " " v_1 v_2 of T.

$$E = \begin{pmatrix} \uparrow & \uparrow \\ v_1 & v_2 \\ \downarrow & \downarrow \end{pmatrix} \qquad D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

$$\& \quad T^n = E \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} E^{-1}.$$

<u>STEP I</u>: Solve $|T - \lambda I| = 0$ for λ

ie $$\left| \begin{pmatrix} 3/4 - \lambda & 1/3 \\ \frac{1}{4} & 2/3 - \lambda \end{pmatrix} \right| = 0$$

$$(3/4 - \lambda)(2/3 - \lambda) - 1/12 = 0$$

ie $$\lambda^2 - \frac{17}{12}\lambda + 5/12 = 0$$

$$(\lambda - 1)(\lambda - 5/12) = 0$$

so $\lambda_1 = 1 \qquad \lambda_2 = 5/12.$

Now Find the eigenvectors

$\lambda_1 = 1$ Solve $(T - 1I)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} -1/4 & 1/3 \\ \frac{1}{4} & -1/3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad \frac{1}{3}y = 1/4\, x \quad \underset{(\times 12)}{\Rightarrow} \quad 4y = 3x$

eg. $v_1 = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

& $\lambda_2 = 5/12$: Solve $\left(T - 5/12\, I\right)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} 3/4 - 5/12 & 1/3 \\ \frac{1}{4} & 2/3 - 5/12 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \quad 1/3\, x + 1/3\, y = 0 \quad \Rightarrow \quad x = -y$

eg. $v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

So $D := \begin{pmatrix} 1 & 0 \\ 0 & 5/12 \end{pmatrix} \Rightarrow D^n = \begin{pmatrix} 1 & 0 \\ 0 & (5/12)^n \end{pmatrix}$

& $E = \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix}$

& $T^n = E D^n E^{-1}$

$$T^n = \underbrace{\frac{1}{7}}_{\frac{1}{|E|}} \underbrace{\begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & \left(\frac{5}{12}\right)^n \end{pmatrix}}_{D^n} \underbrace{\begin{pmatrix} 1 & 1 \\ -3 & 4 \end{pmatrix}}_{E^{-1}}$$

So

$$= \frac{1}{7} \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ (-3)\left(\frac{5}{12}\right)^n & (4)\left(\frac{5}{12}\right)^n \end{pmatrix}$$

as $n \to \infty$

$$= \frac{1}{7} \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

So as $n \to \infty$

$$\begin{pmatrix} f_n \\ g_n \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 4 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_0 \\ g_0 \end{pmatrix}$$

$$= \frac{1}{7} \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} f_0 \\ g_0 \end{pmatrix}$$

$$= \frac{1}{7} \begin{pmatrix} 4(\overbrace{f_0 + g_0}^{=1}) \\ 3(f_0 + g_0) \end{pmatrix}$$

$$= \frac{1}{7} \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$ is the steady state.

= The Normalised eigen vector for $\lambda = 1$