Least Squares

1. $Y = \begin{pmatrix} 2 \\ 0 \\ -3 \\ -5 \end{pmatrix} \qquad X = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$

Need to find

$$y = b_0 + b_1 x$$

$$\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = B = (X^t X)^{-1} X^t Y$$

$$B = \left[\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \right]^{-1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -3 \\ 5 \end{pmatrix}$$

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2. $Y = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \qquad X = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{pmatrix}$

Need to find $y = b_0 + b_1 x$ where

$$\begin{pmatrix} b_0 \\ b_1 \end{pmatrix} = (X^t X)^{-1} X^t Y$$

Orthogonal matrix :

$$A A^t = I$$

Principal Component Analysis

4. $f(x,y) = x^2 + 4xy + 4y^2$

$$(x, y) \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + 4xy + 4y^2$$

$$0 = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) - 4$$

$$= \lambda^2 - 5\lambda + 4 - 4$$

$$= \lambda(\lambda - 5)$$

$\lambda = 0, 5$

$$\lambda = 0 \quad \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\lambda = 5 \quad \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$f(2,-1) = 0$

$f(1, 2) = 25$

Max value of f on S^1 is 25.

$$E = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \qquad EE^T = I$$

$$E A E^{-1} = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$$

Find $\phi : \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (x', y')$

such that

$$f\phi(x,y) = \lambda_1 x'^2 + \lambda_2 x'^2$$

$$f(x,y) = (x,y) \, A \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= (x,y) \, E^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} E \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \left(E\begin{pmatrix} x \\ y \end{pmatrix}\right)^t \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix} E\begin{pmatrix} x \\ y \end{pmatrix}$$

Define $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

which gives the required

transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$.