Defn A simplicial complex consists of a set V and a collection K of certain subsets $\sigma \subseteq V$. The following axioms must hold:

1) $\{v\}$ is in the collection K for all $v \in V$.
2) If $\sigma \in K$ and if $\sigma' \subset \sigma$ then $\sigma' \in K$.

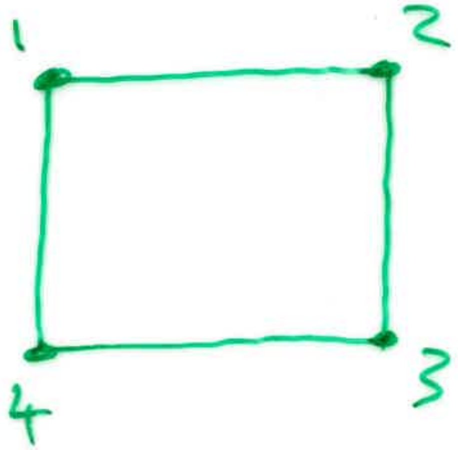
We call the elements $v \in V$ vertices, and the subsets $\sigma \in K$ simplices. We call σ an n-simplex if $\sigma = \{v_0, v_1, \dots, v_n\}$ consists of $n+1$ vertices.

Example $V = \{1, 2, 3, 4\}$

$$K = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\}$$

In this example we just have 0-simplices and 1-simplices.

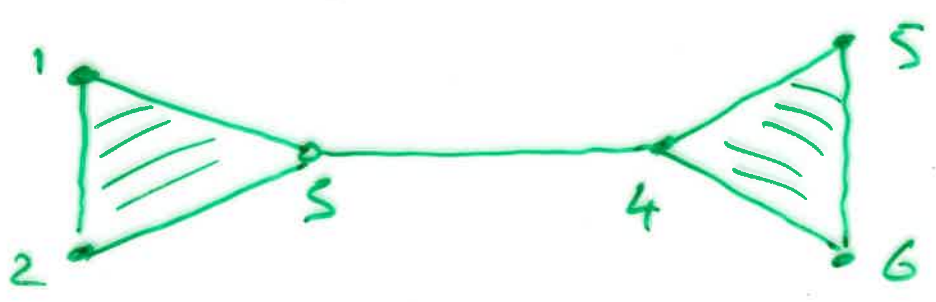
We'll call 0-simplices <u>vertices</u>, and 1-simplices <u>edges</u>. We can picture this K as a graph



<u>Example</u> $V = \{1,2,3,4,5,6\}$

$$K = \{ \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{4,5\}, \{4,6\}, \{5,6\}, \{1,2,3\}, \{4,5,6\} \}$$

We have six 0-simplices, seven 1-simplices and two 2-simplices. We can picture K as



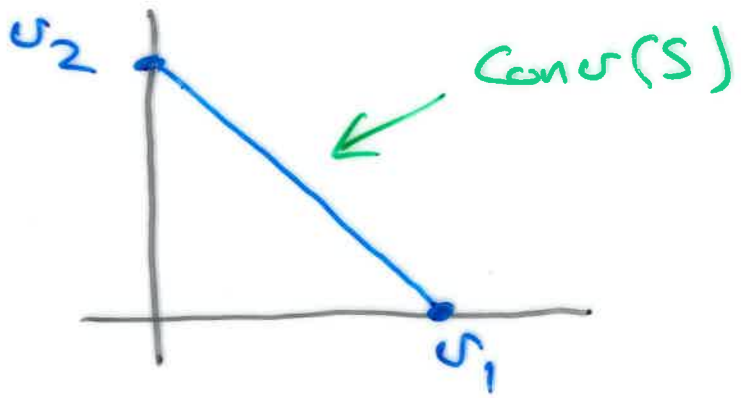
Defn Given a set $S = \{v_1, \dots, v_k\} \in \mathbb{R}^n$ of vectors in $\mathbb{R}^n$, we define its

convex hull

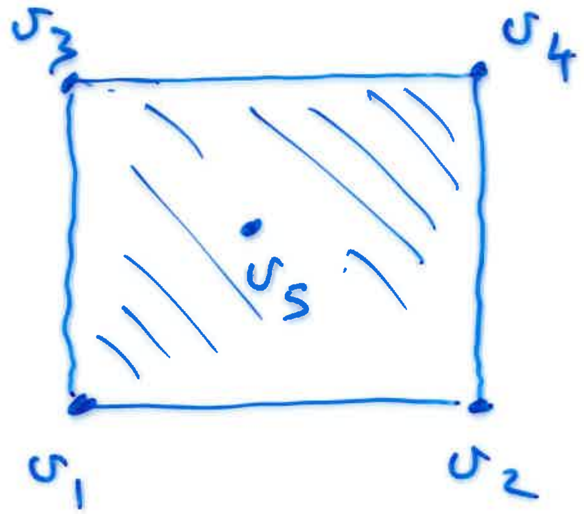
$$\mathrm{Conv}(S) = \left\{ w = \lambda_1 v_1 + \dots + \lambda_k v_k : \lambda_i \geq 0 \text{ for all } i, \text{ and } \sum_{i=1}^{k} \lambda_i = 1 \right\}$$

Example $S = \{v_1 = (1,0),\ v_2 = (0,1)\} \subseteq \mathbb{R}^2$.

$$\mathrm{Conv}(S) = \{\lambda_1 v_1 + \lambda_2 v_2 : \lambda_1, \lambda_2 \geq 0,\ \lambda_1 + \lambda_2 = 1\}$$



Example $S = \{v_1 = (0,0),\ v_2 = (1,0),\ v_3 = (0,1),\ v_4 = (1,1),\ v_5 = (\frac{1}{2}, \frac{1}{2})\}$
$\subseteq \mathbb{R}^2$.



Conv(S) is a square region in $\mathbb{R}^2$.

<u>Defn</u> Let $v_0, v_1, \dots, v_k \in \mathbb{R}^d$ be $k+1$ vectors such that the k vectors

$$v_1 - v_0, \ v_2 - v_0, \ \dots, \ v_k - v_0$$

are linearly independent. We say

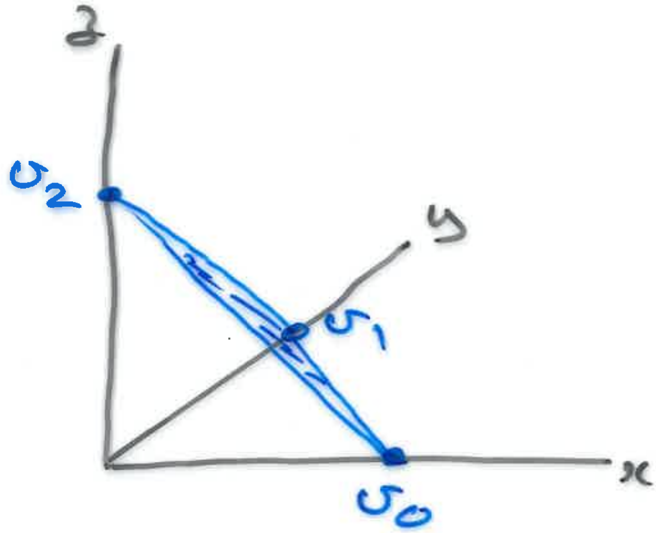
$$\mathrm{Conv}(\{v_0, v_1, \dots, v_k\})$$

is a <u>geometric</u> <u>k-simplex</u>.

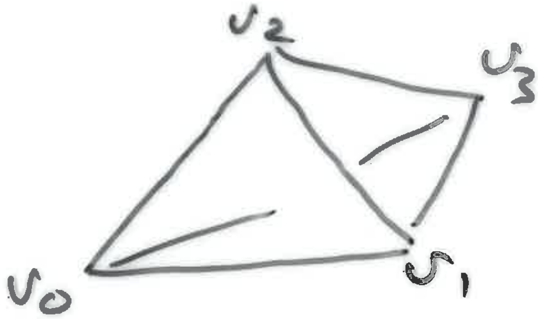
<u>Example</u> $S = \{ v_0 = (1,0,0), \ v_1 = (0,1,0), \ v_2 = (0,0,1) \} \subseteq \mathbb{R}^3$.

$v_1 - v_0, \ v_2 - v_0$ are linearly independent.

So Conv(S) is a geometric 2-simplex.



Example Geometric 3-simplex



Solid tetrahedron

Suppose (K, V) is a simplicial complex with finite vertex set

$V = \{v_1, v_2, \dots, v_n\}$ say.

Now identify v_i with the i^{th} standard basis vector of $\mathbb{R}^n$,

$$v_i = (0 \cdots 0, 1, 0 \cdots 0),$$

position i

For each k-simplex

$$\sigma = \{v_{i_0}, v_{i_1}, \dots, v_{i_k}\}$$

we let

$$|\sigma| = \mathrm{Conv}(\{v_{i_0}, \dots, v_{i_k}\})$$

denote the corresponding geometric k-simplex.

Defn The *geometric realisation* $|K|$ is the subset of $\mathbb{R}^n$ arising as the union of the geometric simplices $|\sigma|$ with $\sigma \in K$.

Note that $|K|$ is a topological subspace of $\mathbb{R}^n$.

Suppose given an $n \times n$ distance matrix $D = (d_{ij})$, recording dissimilarities between n items

For any $\varepsilon > 0$ we can construct a simplicial complex (K_ε, V) with

$$V = \{1, 2, \ldots, n\}$$

$$K_\varepsilon = \{\sigma \subseteq V : d_{ij} \leq \varepsilon \text{ for all } i,j \in \sigma\}$$

We can now associate the topological space $|K_\varepsilon|$.