Recall: finite set $S \subseteq \mathbb{R}^n$

$p \in S$

$$V(p) = \{ v \in \mathbb{R}^n : \|p - v\| \le \|p' - v\| \text{ for all } p' \in S \}$$

Recall: $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$

$$u \cdot v = u v^t$$

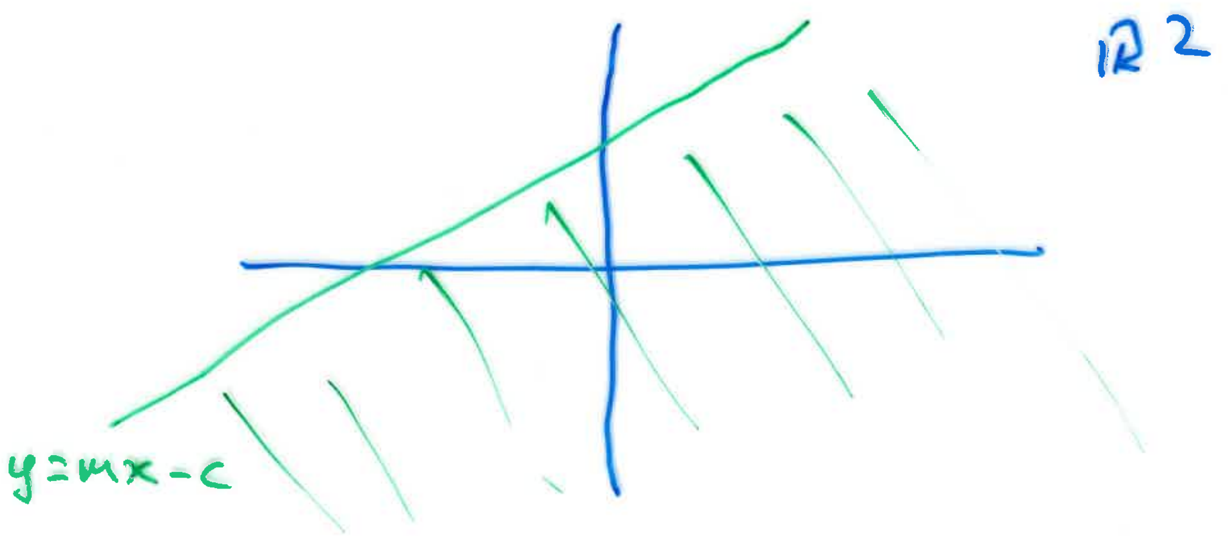
Defn For $w \in \mathbb{R}^n$ and any real number c we define the half-space

$$H(w, c) = \{ v \in \mathbb{R}^n : w \cdot v \ge c \}$$

Example For $w = (m, -1) \in \mathbb{R}^2$, $c \in \mathbb{R}$

$$H(w, c) = \{ (x, y) \in \mathbb{R}^2 : (x, y) \cdot (m, -1) \ge c \}$$

$$= \{ (x, y) \in \mathbb{R}^2 : y \le mx - c \}$$



Theorem The Voronoi region $V(p)$ is an intersection of finitely many half-spaces

$$V(p) = H(\omega_1, c_1) \cap H(\omega_2, c_2) \cap \dots \cap H(\omega_k, c_k).$$

In this theorem we can choose the half-spaces such that

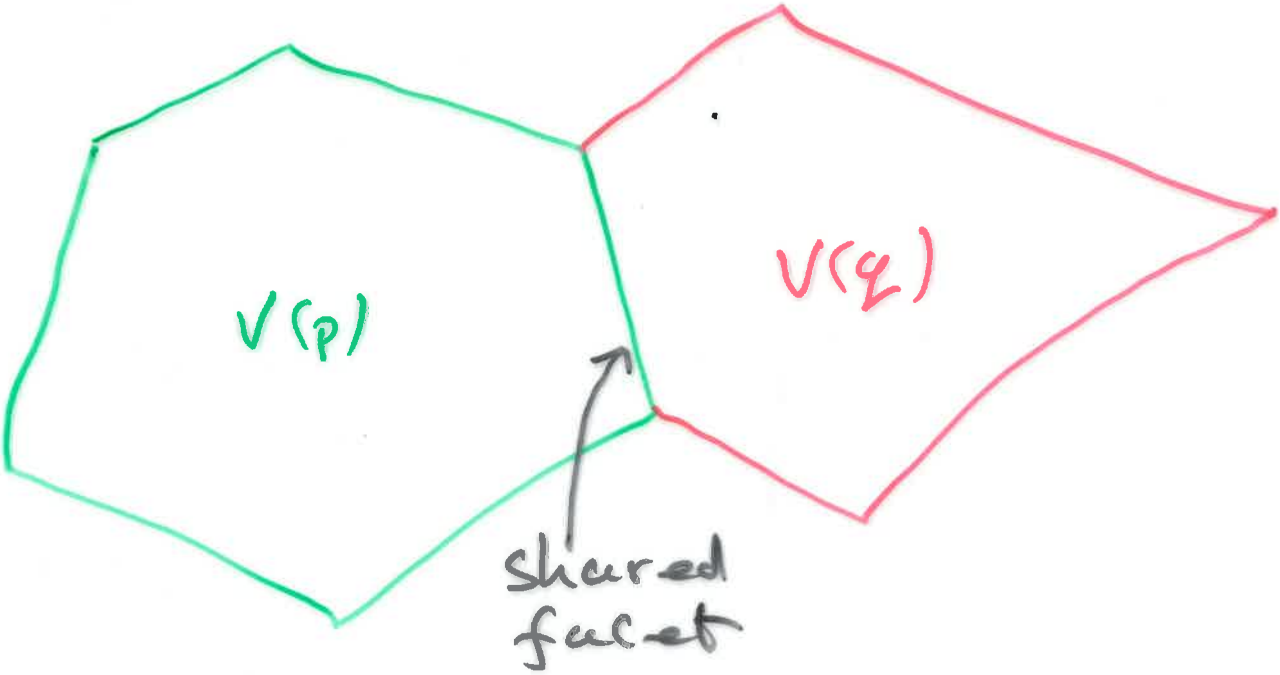
$$F_i = V(p) \cap H(-\omega_i, c_i)$$

is an $n-1$-dimensional subset of $\mathbb{R}^n$ (i.e. contains n vectors $v_0, v_1, \dots, v_{n-1} \in \mathbb{R}^n$ such that

$$v_1 - v_0,\ v_2 - v_0,\ \dots,\ v_{n-1} - v_0$$

are linearly independent) These F_i are called facets.

Defn We say that points $p, q \in S$ are neighbours if their regions $V(p), V(q)$ share a facet.



for our database of points $S \subset \mathbb{R}^n$
we compute $V(p)$ for each $p \in S$
and then record for each p
the neighbours

$$N(p) = \left\{ q \in S : q \neq p \text{ and } V(p) \text{ shares a facet with } V(q) \right\}$$

Algorithm

Input: $S \subset \mathbb{R}^n$ and $v \in \mathbb{R}^n$

Output: The point p in S closest to v.

procedure

1. Choose some $p_0 \in S$
2. while $\exists$ $q \in N(p_0)$ with $\|q - v\| < \|p_0 - v\|$ do:

   set $p_0 := q$

   end do;

Theorem (Johnson-Lindenstrauss)

Let $\varepsilon \in (0, \frac{1}{2})$. Let $S \subseteq \mathbb{R}^d$ be a set of n points. Set

$$k = \frac{20 \log(n)}{\varepsilon^2}$$

There exists a linear hom^{sm} $f : \mathbb{R}^d \longrightarrow \mathbb{R}^k$ such that for all $u, v \in S$

$$(1-\varepsilon)\|u-v\|^2 \leq \|f(u)-f(v)\|^2 \leq (1+\varepsilon)\|u-v\|^2 .$$

Example If $S \subseteq \mathbb{R}^{1000000}$ has $n = |S| = 10000$ and if $\varepsilon = 0.499$ then $k = 20 \cdot 4 \cdot 4 = 320$

$$f : \mathbb{R}^{1000000} \longrightarrow \mathbb{R}^{320}$$

$$0.71\|u-v\| \leq \|f(u)-f(v)\| \leq 1.23\|u-v\|$$

To prove the J-L Theorem we need:

<u>Proposition</u> (Norm preservation)

Let $x \in \mathbb{R}^d$. Assume that A is a $k \times d$ matrix whose kd are sampled independently from $N(0,1)$. Then

$$\Pr\left((1-\varepsilon)\|x\|^2 \leq \|\tfrac{1}{k} A x\| \leq (1+\varepsilon)\|x^2\|\right)$$

$$\geq 1 - 2e^{-(\varepsilon^2 - \varepsilon^3)k/4}.$$