

## MA342: Tutorial Problems 2018-19

Tutorials: Wednesday, 2-3pm, Venue = AC201  
Wednesday, 6-7pm, Venue = AM150

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### PROBLEMS

## 1 Euler characteristics

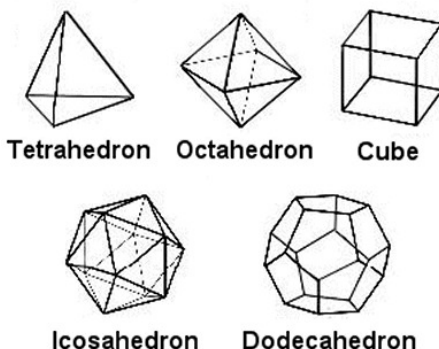
1. Draw a graph on a sphere  $\mathbb{S}^2$



in such a way that if two edges intersect then they intersect in a vertex of the graph. Determine the number of vertices  $V$ , edges  $E$  and faces  $F$  for your graph. Then compute the Euler characteristic  $\chi(\mathbb{S}^2) = V - E + F$ .

2. Prove that the value of the Euler characteristic  $\chi(\mathbb{S}^2) = V - E + F$  in Problem 1 does not depend on your particular choice of graph on the sphere. [See Lecture 1.]
3. A *platonic solid* is a 3-dimensional convex object whose surface is the union of a finite number of polygonal planar faces satisfying:
  - (a) all faces are congruent to some fixed regular  $p$ -gon;
  - (b) the intersection of two faces is either empty or a common edge of the two faces or a common vertex of the two faces;
  - (c) the same number of faces,  $q$ , meet at each vertex.

Five platonic solids are shown in the following figure.



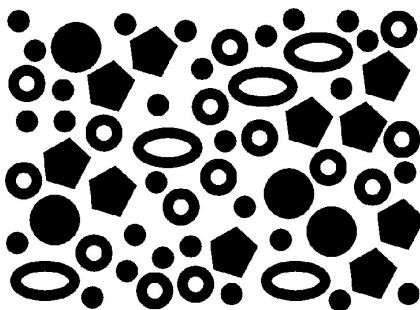
Use the Euler characteristic to prove

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}, \quad e \geq 0$$

for any platonic solid.

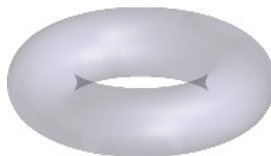
4. Deduce from Problem 3 that there are only five platonic solids.

5. The digital image



represents a region  $X \subseteq \mathbb{R}^2$  formed as a union of various unit squares  $[m, m + 1] \times [n, n + 1]$  for various integers  $m, n$ . Determine  $\chi(X)$ .

6. Draw a graph on a torus  $\mathbb{T}$



in such a way that

- (a) if two edges of the graph intersect then they intersect in a vertex of the graph;
- (b) each resulting face on the torus is a curvilinear disk (*i.e.* a “continuous deformation” of some planar polygonal disk).

Determine the number of vertices  $V$ , edges  $E$  and faces  $F$  for your graph. Then compute the Euler characteristic  $\chi(\mathbb{T}) = V - E + F$ . [The term “continuous deformation” will be made precise later in the course: it is just a *homeomorphism*.]

- 7. Prove that the value of the Euler characteristic  $\chi(\mathbb{T}) = V - E + F$  in Problem 6 does not depend on your particular choice of graph on the torus. [Hint: The torus  $\mathbb{T}$  can be constructed from a rectangular sheet of paper by identifying/gluing opposite sides of the sheet. We know that the Euler characteristic of a solid plane rectangle is 1.]
- 8. A *polygonal surface* is a union of curvilinear polygonal disks such that, if two polygonal disks intersect, then their intersection is a union of edges and/or vertices of the disks. The polygonal disks are called *faces*. The soccer ball is an example of a

polygonal surface with pentagonal and hexagonal faces. For any polygonal surface  $X$  we define the *Euler characteristic*  $\chi(X) = V - E + F$  where  $V$  is the number of vertices on  $X$ ,  $E$  the number of edges and  $F$  the number of faces.

Suppose that  $X$  is a polygonal surface. Let  $A, B \subseteq X$  be subsets of  $X$  each arising as a union of faces. Prove that

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

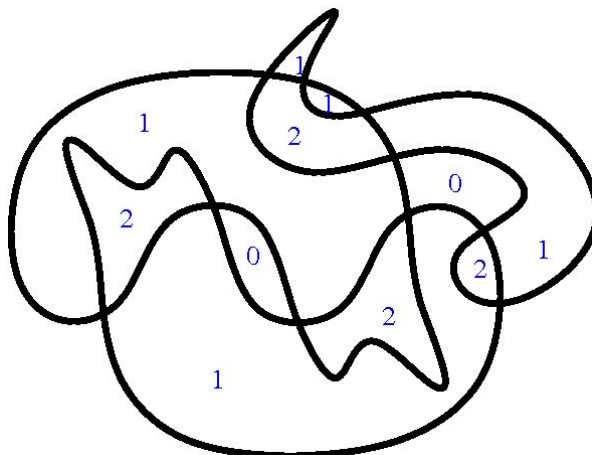
[See Lecture 2.]

- Use the formula  $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$  to determine the Euler characteristic of a double torus surface.



## 2 Euler Integration

- The following picture shows the boundaries of several regions  $U_1, \dots, U_t \subseteq \mathbb{R}^2$  of common Euler characteristic  $\chi(U_i) = 1$ .



No two boundaries are tangential at any point. The numbers in the interiors of the regions and their intersections represent the weight function  $\omega: X \rightarrow \mathbb{N}$  where  $X = U_1 \cup U_2 \cup \dots \cup U_t$  and  $w(x) = |\{i : x \in U_i\}|$ . Evaluate the Euler integral

$$\int_X \omega d\chi$$

and then determine the number of regions  $t$ .

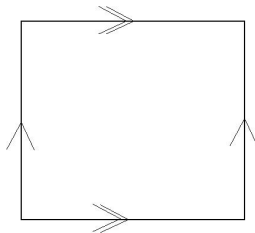
- Let  $X \subseteq \mathbb{R}^2$  be a region arising as the union of subregions  $U_1, U_2, \dots, U_t \subseteq X$  of common Euler characteristic  $\chi(U_i) = C$ . Let  $\omega: X \rightarrow \mathbb{N}$  be the weight function given by  $w(x) = |\{i : x \in U_i\}|$ . Prove that

$$t = \frac{1}{C} \int_X \omega d\chi .$$

[See Lecture 4.]

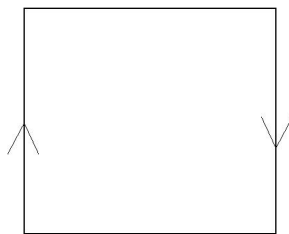
### 3 Möbius strips, Klein bottles ...

- The torus  $\mathbb{T}$  is obtained from the unit square  $[0, 1] \times [0, 1]$  by making the identifications  $(x, 0) = (x, 1)$  and  $(0, y) = (1, y)$  for  $x, y \in [0, 1]$ .



Is it true that any loop in  $\mathbb{T}$  that has no self intersections cuts  $\mathbb{T}$  into two components? If not, then exhibit a loop that does not cut  $\mathbb{T}$  into two components. [See Lecture 4]

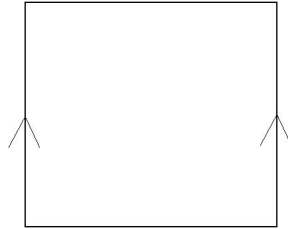
- The Möbius strip  $\mathbb{M}$  is obtained from the unit square  $[0, 1] \times [0, 1]$  by making the identifications  $(0, y) = (1, 1 - y)$  for  $y \in [0, 1]$ .



Is it true that any loop in  $\mathbb{M}$  that has no self intersections cuts  $\mathbb{M}$  into two components? If not, then exhibit a loop that does not cut  $\mathbb{M}$  into two components. [See Lecture 4]

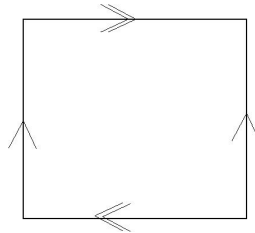
- The cylinder  $X$  is obtained from the unit square  $[0, 1] \times [0, 1]$  by making the identi-

fications  $(0, y) = (1, y)$  for  $y \in [0, 1]$ .



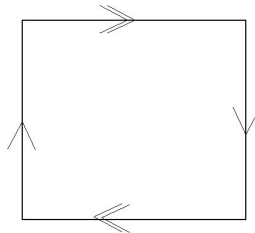
Is it true that any loop in  $X$  that has no self intersections cuts  $X$  into two components? If not, then exhibit a loop that does not cut  $X$  into two components.

4. The Klein bottle  $\mathbb{K}$  is obtained from the unit square  $[0, 1] \times [0, 1]$  by making the identifications  $(x, 0) = (1 - x, 1)$  and  $(0, y) = (1, y)$  for  $x, y \in [0, 1]$ .



Is it true that any loop in  $\mathbb{K}$  that has no self intersections cuts  $\mathbb{K}$  into two components? If not, then exhibit a loop that does not cut  $\mathbb{K}$  into two components.

5. The projective plane  $\mathbb{P}$  is obtained from the unit square  $[0, 1] \times [0, 1]$  by making the identifications  $(x, 0) = (1 - x, 1)$  and  $(0, y) = (1, 1 - y)$  for  $x, y \in [0, 1]$ .



Is it true that any loop in  $\mathbb{P}$  that has no self intersections cuts  $\mathbb{P}$  into two components? If not, then exhibit a loop that does not cut  $\mathbb{P}$  into two components.

## 4 Subsets of Euclidean space

1. Exhibit a collection of open subsets of the plane  $\mathbb{E}^2$  whose intersection is not open. [Lecture 5]
2. Let  $X \subset \mathbb{E}^2$  be the set of those points in the plane that have at least one rational coordinate. Is  $X$  an open subset of  $\mathbb{E}^2$ ? Is  $X$  a connected subset of  $\mathbb{E}^2$ ? Justify your answers.

3. Let

$$\begin{aligned} Y &= \{(0, y) \in \mathbb{E}^2 : -1 < y < 1\}, \\ Z &= \{(x, \sin(1/x)) \in \mathbb{E}^2 : 0 < x \leq 1\}, \\ X &= Y \cup Z. \end{aligned}$$

Is  $X$  an open subset of  $\mathbb{E}^2$ ? Is  $X$  a connected subset of  $\mathbb{E}^2$ ? Justify your answers.

## 5 Topological spaces

- For each of the following sets  $X$  and collections  $T$  of open subsets decide if the pair  $X, T$  satisfies the axioms of a topological space. If it does, determine whether  $X$  is connected. If it is not a topological space then explain which axioms fail.
  - $X = \mathbb{R}^n$  and the subset  $U \subset X$  is open if, for any  $x \in U$ , there is a real  $\epsilon > 0$  such that the open Euclidean ball  $B^n(x, \epsilon)$  of radius  $\epsilon$  and centred at  $x$  is contained in  $U$ .
  - $X = \mathbb{R}^n$  and the subset  $U \subset X$  is open if, for any  $x \in X \setminus U$ , there is a real  $\epsilon > 0$  such that the open Euclidean ball  $B^n(x, \epsilon)$  of radius  $\epsilon$  and centred at  $x$  is contained in the complement  $X \setminus U$ .
  - $X = \mathbb{R}^n$  and every subset  $U \subset X$  is open.
  - $X = \mathbb{R}^n$  and the only open subsets are  $X$  and the empty set  $\emptyset$ .
  - $X = \mathbb{Z}$  and a subset  $U \subset \mathbb{Z}$  is open if and only if its complement  $\mathbb{Z} \setminus U$  is finite or  $U = \emptyset$ .
  - $X = \mathbb{Z}$  and a subset  $U \subset \mathbb{Z}$  is open if and only if  $U$  is finite or  $U = \mathbb{Z}$ .
  - $X = \mathbb{R}^n$  and a subset  $U \subset \mathbb{R}^n$  is open if and only if it is a vector subspace of  $\mathbb{R}^n$ . Here  $\mathbb{R}^n$  has the standard addition and scalar multiplication.
  - $X = \{1, 2, 3, 4\}$  and  $T = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$ .
  - $X = \mathbb{Z}$  and a subset  $U \subset \mathbb{Z}$  is open if and only if each of its elements is even.

## 6 Subspaces

- Given a topological space  $X$ , define what it means for a subset  $Y \subseteq X$  to be a *subspace*.
- For each of the following topological spaces  $X$  and subspaces  $Y \subseteq X$  describe the connected components of  $Y$ .
  - $X = \mathbb{E}^2, Y = \{(x, y) \in \mathbb{E}^2 : x^2 + y^2 \neq 1\}$ .
  - $X = \mathbb{E}^1, Y = \mathbb{Q}$ .

- (c)  $X = \mathbb{E}^2, Y = \{(x, y) \in \mathbb{E}^2 : x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$ .  
 (d)  $X = \mathbb{E}^2, Y = \{(x, y) \in \mathbb{E}^2 : x \in \mathbb{Z} \text{ and } y \in \mathbb{Z}\}$ .

3. The table

	<i>H</i>	<i>M</i>	<i>R</i>	<i>C</i>	<i>W</i>
<i>H</i>	0	11	10	14	22
<i>M</i>	11	0	3	13	21
<i>R</i>	10	3	0	12	20
<i>C</i>	14	13	12	0	16
<i>W</i>	22	21	20	16	0

gives distances between the species Human, Mouse, Rat, Cat, Whale. For  $\epsilon > 0$  let  $G_\epsilon$  denote the graph with vertices  $H, M, R, C, W$  and with an edge between vertices  $X$  and  $Y$  if  $dist(X, Y) \leq \epsilon$ .

- (a) Sketch the graphs  $G_4, G_{10}, G_{16}$ .  
 (b) Explain how one can view the graphs  $G_\epsilon$  as subspaces of  $\mathbb{E}^5$ .  
 (c) Draw the dendrogram that describes the inclusion relationships between the connected components of the subspaces  $G_0, G_2, G_4, \dots, G_{18}, G_{20}$ .

## 7 Some useful jargon

1. Let  $X$  be a topological space and let  $W \subseteq X$  be some subset.
  - The subset  $W$  is said to be *closed* if the complement  $X \setminus W$  is an open subset of  $X$ .
  - A point  $x \in X$  is said to be a *limit point* of  $W$  if every open set  $U \subset X$  containing  $x$  has non-empty intersection with  $W$ .
  - The union of  $W$  and all its limit points in  $X$  is said to be the *closure* of  $W$ . The closure is denoted by  $\overline{W}$ .
  - (a) Prove that the closure  $\overline{W}$  is a closed subset of  $X$ .
  - (b) Suppose that  $W \subseteq Z$  where  $Z$  is a closed subset of  $X$ . Prove that  $\overline{W} \subseteq Z$ .
  - (c) Prove that  $\overline{W}$  is equal to the intersection of all closed subsets of  $X$  containing  $W$ .
  - (d) Prove that a subset  $W$  is closed if and only if  $W = \overline{W}$ .
2. Find a family of closed subsets of the real line whose union is not closed.
3. Describe the closure of the subspace  $W = \{(1/n) \sin(n) : n = 1, 2, \dots\}$  of the real line.
4. Let  $Y$  be a subspace of  $X$ . Show that if  $A$  is closed in  $Y$  and if  $Y$  is closed in  $X$  then  $A$  is closed in  $X$ .

## 8 Continuity

1. Give the definition of a continuous function between topological spaces.
2. Let  $\mathbb{R}$  denote the real line with its usual topology. Is the function  $f: \mathbb{R} \rightarrow \mathbb{R}$  given by

$$f(x) = \begin{cases} x^2, & x \leq 1, \\ x^2 + 2, & x > 1 \end{cases}$$

continuous? Justify your answer.

3. Let  $\mathbb{R}$  denote the real line with its usual topology. Let  $X = \mathbb{R} \setminus \{1\}$  be given the subspace topology. Is the function  $f: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R} \setminus \{1\}$  given by

$$f(x) = \begin{cases} x^2, & x < 1, \\ x^2 + 2, & x > 1 \end{cases}$$

continuous? Justify your answer.

4. Suppose that  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  are continuous functions. Prove that the composite  $gf: X \rightarrow Z, x \mapsto g(f(x))$  is continuous.
5. Suppose that  $f: X \rightarrow Y$  and  $g: Y \rightarrow Z$  are homeomorphisms. Prove that the composite  $gf: X \rightarrow Z, x \mapsto g(f(x))$  is a homeomorphism.
6. Consider the set  $X = \{a, b, c\}$  endowed with the topology  $T_X = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$ , and the set  $Y = \{a, b, c, d\}$  endowed with the topology  $T_Y = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$ . Is the function  $f: X \rightarrow Y, a \mapsto a, b \mapsto b, c \mapsto c$  continuous?
7. Let  $\mathbb{Z}$  denote the integers endowed with the cofinite topology. Exhibit an example of a function  $f: \mathbb{Z} \rightarrow \mathbb{Z}$  which is not continuous.
4. Prove that the unit circle  $S^1 = \{(x, y) \in \mathbb{E}^2 : x^2 + y^2 = 1\}$  is homeomorphic to the square  $Y = \{(x, y) \in \mathbb{E}^2 : -2 \leq x, y \leq 2, \text{ and either } x \in \{-2, 2\} \text{ or } y \in \{-2, 2\}\}$ .
5. Let  $\Delta$  denote an equilateral triangular region in the plane  $\mathbb{E}^2$ . Describe the construction of a continuous surjective function  $f: [0, 1] \rightarrow \Delta$ . (You are *not* asked to prove any convergence nor to prove surjectivity.)
6. Prove that if  $X$  and  $Y$  are homeomorphic then  $X$  is connected if and only if  $Y$  is connected.
7. Prove that  $(0, 1)$  is homeomorphic to  $\mathbb{E}$ .
8. Prove that  $\mathbb{E}$  is not homeomorphic to  $\mathbb{E}^2$ .
9. Prove that  $\mathbb{E}$  is not homeomorphic to the space  $Y = \{(x, y) \in \mathbb{E}^2 : x = 0 \text{ or } y = 0\}$ . (Here  $Y$  is the union of the  $x$ -axis and  $y$ -axis.)

## 9 Compactness

1. Explain why  $\mathbb{E}^2$  is not compact.
2. Prove that the interval  $[0, 1]$  is compact.
3. Prove that if  $X$  is compact and  $f: X \rightarrow Y$  is a continuous map then the image of  $f$  is compact.
4. Prove that if  $X$  is compact and  $Y$  is homeomorphic to  $X$  then  $Y$  is compact.
5. Determine the accumulation points of the subset  $A = \{1/n\}_{n=1,2,3,\dots}$  of  $\mathbb{R}$ .
6. Determine the accumulation points of the subset  $A = (0, 1)$  of  $\mathbb{R}$ .
7. Prove that a subset  $A$  of a topological space  $X$  is closed if and only if it contains all its accumulation points.
8. Describe a surjective continuous map  $f: [0, 1] \rightarrow \Delta$  where  $\Delta \subset \mathbb{E}^2$  is a solid equilateral triangle. Explain why  $f$  is surjective, stating clearly any theorems that you use in your explanation.
9. Prove that if  $Y$  is homeomorphic to a Hausdorff topological space  $X$  then  $Y$  is Hausdorff.
10. Let  $\mathbb{Z}$  be endowed with the cofinite topology. Is  $\mathbb{Z}$  Hausdorff with this topology? Exhibit a topology on  $\mathbb{Z}$  for which the space is Hausdorff.
11. Prove that a compact subset  $A$  of a Hausdorff space  $X$  is closed.

The second in-class test will consist of five few of the above questions.

## 10 Simplicial complexes

1. Describe a triangulation on the torus. Determine the number of  $k$ -simplices in your triangulation for  $k = 0, 1, 2$  and then compute the Euler characteristic.
2. Determine the Euler characteristic of the Möbius band.
3. Determine the Euler characteristic of the sphere  $S^n = \{x \in \mathbb{E}^{n+1} : \|x\| = 1\}$  for  $n = 1, 2, 3, \dots$

- Describe a triangulation on the double torus.



Determine the number of  $k$ -simplices in your triangulation for  $k = 0, 1, 2$  and then compute the Euler characteristic.

## 11 Homotopy of maps

- Let  $Y \subset \mathbb{E}^n$  be an arbitrary convex subset of Euclidean space and let  $X$  be an arbitrary topological space. Prove that any two continuous maps  $f, g: X \rightarrow Y$  are homotopy equivalent.
- Prove that homotopy equivalence of maps  $f \simeq g$  is an equivalence relation on the set of continuous maps  $X \rightarrow Y$  from a given space  $X$  to a given space  $Y$ .
- Let  $f: [0, 1] \rightarrow S^1$  be a continuous map. Prove that there is a unique continuous map  $\tilde{f}: [0, 1] \rightarrow \mathbb{R}$  such that  $f(t) = e(i\tilde{f}(t))$  where  $e: \mathbb{R} \rightarrow S^1, \theta \mapsto e^{2\pi\theta i}$ .
- Define the *winding number* of a map  $f: S^1 \rightarrow S^1$  with. Explain why homotopic maps  $f \simeq g$  have the same winding number.
- Let  $[S^1, S^1]$  denote the set of homotopy classes of maps  $S^1 \rightarrow S^1$ . Describe a bijection  $\omega: [S^1, S^1] \xrightarrow{\cong} \mathbb{Z}$ . Explain why  $\omega$  is onto. Explain why  $\omega$  is injective.
- State and prove the Fundamental Theorem of Algebra.

## 12 Homotopy equivalent spaces

- Prove that any convex subspace  $Y \subset \mathbb{E}^n$  is homotopy equivalent to the space consisting of a single point.
- Prove that the complex plane minus the origin  $\mathbb{C} \setminus \{0\}$  is homotopy equivalent to the circle  $S^1$ .
- Prove that homotopy equivalence of maps  $f \simeq g$  is an equivalence relation on the set of continuous maps  $X \rightarrow Y$  from a given space  $X$  to a given space  $Y$ .

4. Use the fact that the Euler characteristic of a triangulable space is a homotopy invariant to prove Brouwer's fixed point theorem: any continuous map  $D^n \rightarrow D^n$  on the closed disc has at least one fixed point.
5. Prove the Frobenius-Perron Theorem: a real square matrix with positive entries has a positive real eigenvalue and the corresponding eigenvector has positive components.

## 13 Nash Equilibrium

1. Describe what is meant by a *Nash Equilibrium*, explaining any concepts from Game Theory that you use.
2. Use Brouwer's fixed point theorem to prove the existence of a Nash equilibrium in a mixed strategy game.