Recall A game involves

- n players,
- a set S_i of strategies for each player i,
- a pay-off function

$$\nu_i : S_1 \times S_2 \times S_3 \times \dots \times S_n \longrightarrow \mathbb{R}$$

for each player i, $1 \le i \le n$.

Example $n = 2$

$S_1 = \{H, T\}$ $\qquad$ $S_2 = \{H, T\}$

$\nu_1(H,H) = 1$ $\nu_2(H,H) = -1$	$\nu_1(H,T) = -1$ $\nu_2(H,T) = 1$
$\nu_1(T,H) = -1$ $\nu_2(T,H) = 1$	$\nu_1(T,T) = 1$ $\nu_2(T,T) = -1$

Defn A mixed strategy is a choice of probabilities

$p_{i,s}$ = probability that play i plays strategy $s \in S_i$

for $1 \le i \le n$, $s \in S_i$ satisfying

$$p_{i,s} \ge 0, \quad \sum_{s \in S_i} p_{i,s} = 1$$

Notation Suppose $S_i = \{s_1, s_2, \dots, s_k\}$ and set

$$\underline{p}_i = (p_{i,s_1}, p_{i,s_2}, \dots, p_{i,s_k})$$

Define the expected payoff for player i to be the function

$$\mathcal{E}_i(\underline{p}_1, \underline{p}_2, \cdots, \underline{p}_n) = E(\nu_i)$$

$$= \sum_{\substack{x_1 \in S_1 \\ x_2 \in S_2 \\ \vdots \\ x_n \in S_n}} p_{1 x_1}\, p_{2 x_2}\, p_{3 x_3} \cdots p_{n x_n}\, \nu_i(x_1, x_2, \cdots, x_n)$$

A mixed Nash equilibrium occurs if, having played the game, no player benefits by unilaterally changing their mixed strategy (the mixed strategies of all other players remaining fixed).

Theorem (J. Nash) In any game with finitely many players and finite pure strategy sets, there exists at least one Nash equilibrium.

Example For the above 2-player game

$S_1 = \{H, T\}, \quad S_2 = \{H, T\}$

$$\Sigma_i(\underline{P}_1, \underline{P}_2) =$$

$$P_{1H} P_{2H} \nu_i(H, H) + P_{1T} P_{2H} \nu_i(T, H) + P_{1H} P_{2T} \nu_i(H, T) + P_{1T} P_{2T} \nu_i(T, T)$$

$$\Sigma_1 = P_{1H} P_{2H} - P_{1T} P_{2H} - P_{1H} P_{2T} + P_{1T} P_{2T}$$

$$\Sigma_2 = -\Sigma_1$$

In this 2-player game an example of a mixed Nash equilibrium is the mixed strategy

$p_{1H} = \frac{1}{2}$, $p_{1T} = \frac{1}{2}$, $p_{2H} = \frac{1}{2}$, $p_{2T} = \frac{1}{2}$

Outline proof

Consider

$$C = \left\{ (\underline{p}_1, \underline{p}_2, \dots, \underline{p}_n) \right\} \subseteq \mathbb{R}^{|S_1| + |S_2| + \cdots + |S_n|}$$

where $\underline{p}_i \in \mathbb{R}^{|S_i|}$ is the probability distribution for player i.

Now $p_{i,s} \geq 0$, $\sum_{s \in S_i} p_{i,s} = 1$ means that C is closed, bounded and convex.

Thus, by Brouwer's Theorem, any continuous function

$$f : C \longrightarrow C$$

has at least one fixed point.

For a given $(\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n) \in C$ define $\underline{q}_i \in \mathbb{R}^{|S_i|}$ to be the probability distribution that maximizes

$$E_i(\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_{i-1}, \underline{q}_i, \underline{p}_{i+1}, \ldots, \underline{p}_n) \qquad (*)$$

Now define $f : C \to C$ by

$$f(\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n) = (\underline{q}_1, \underline{q}_2, \ldots, \underline{q}_n).$$

This f has a fixed point. But this fixed point is a mixed Nash equilibrium. $\square$

Slight problem: The quantity $\underline{q}_i$ that maximizes (*) may not be unique. Thus f is not a well-defined function.

To overcome this problem one replaces (*) by

$$E_i(\underline{p}_1, \dots, \underline{p}_{i-1}, \underline{q}_i, \underline{p}_{i+1}, \dots, \underline{p}_n) - \|\underline{p}_i - \underline{q}_i\|^2 \qquad (*)$$