Last time:

$$\pi_1(S^1, 1) \cong \mathbb{Z}$$

## Fundamental theorem of Algebra

Any polynomial

$$a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

with $a_i \in \mathbb{C}$ and of degree $n > 0$ has at least one zero in $\mathbb{C}$.

Proof Since $a_n \neq 0$, a scalar multiple of the polynomial has the form

$$p(z) = z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + \cdots + a_0$$

Let's suppose $p(z) \neq 0$ for all $z \in \mathbb{C}$.

for $\lambda \geq 0$ define

$$f_\lambda : S^1 \longrightarrow S^1$$

by

$$f_\lambda(z) = \frac{p(\lambda z)}{|p(\lambda z)|} .$$

Any two maps $f_\lambda$, $f_{\lambda'}$ are homotopic via the homotopy

$$H_t(z) = \frac{p(((1-t)\lambda + t\lambda')\, z)}{|p((1-t)\lambda + t\lambda')\, z)|}$$

Note: for $\lambda = 0$ we have that $f_0$ is a constant function, and has winding number $0$.

<u>Exercise</u> (tricky):
for large $\lambda$ we have that
$f_\lambda(z)$ is homotopic to

$$g_n : S^1 \to S^1, \quad z \mapsto z^n$$

But $g_n$ has winding number $n$.

But $g_n \simeq f_\lambda \simeq f_0$, and homotopic maps have the same winding number.

Hence the winding number of $g_n$ is $0$.

Contradiction. $\boxed{\mathcal{U}}$

# Game Theory

A game involves

- $n$ players
- a set $S_i$ of strategies for player $i$.
- a payoff function

$$v_i : S_1 \times S_2 \times \cdots \times S_n \longrightarrow \mathbb{R}$$

for each player $i$, $i = 1, 2, \cdots, n$.

Example 1 Two players, Mary and John. They want to go to the cinema (C) or to a soccer match (S) together.

| | |
|---|---|
| $v_1(C,C) = 2$<br>$v_2(C,C) = 1$ | $v_1(C,S) = 0$<br>$v_2(C,S) = 0$ |
| $v_1(S,C) = 0$<br>$v_2(S,C) = 0$ | $v_1(S,S) = 1$<br>$v_2(S,S) = 2$ |

**Example 2** Two players, each places a coin on the table. Player 1 wants coins to be the same, Player 2 wants the coins to be different.

$$S_1 = \{H, T\}$$

$$S_2 = \{H, T\}$$

payoff function:

| | |
|---|---|
| $\nu_1(H,H) = 1$<br>$\nu_2(H,H) = -1$ | $\nu_1(H,T) = -1$<br>$\nu_2(H,T) = 1$ |
| $\nu_1(T,H) = -1$<br>$\nu_2(T,H) = 1$ | $\nu_1(T,T) = 1$<br>$\nu_2(T,T) = -1$. |

In a **pure strategy game** decides beforehand on a strategy to play.

A pure Nash equilibrium

occurs if, having played the game, no player benefits from unilaterally changing his/her choice of strategy.

Example 1 There are two pure Nash equilibria:

- both go to cinema
- both go to soccer.

Example 2 There is no pure Nash equilibrium.