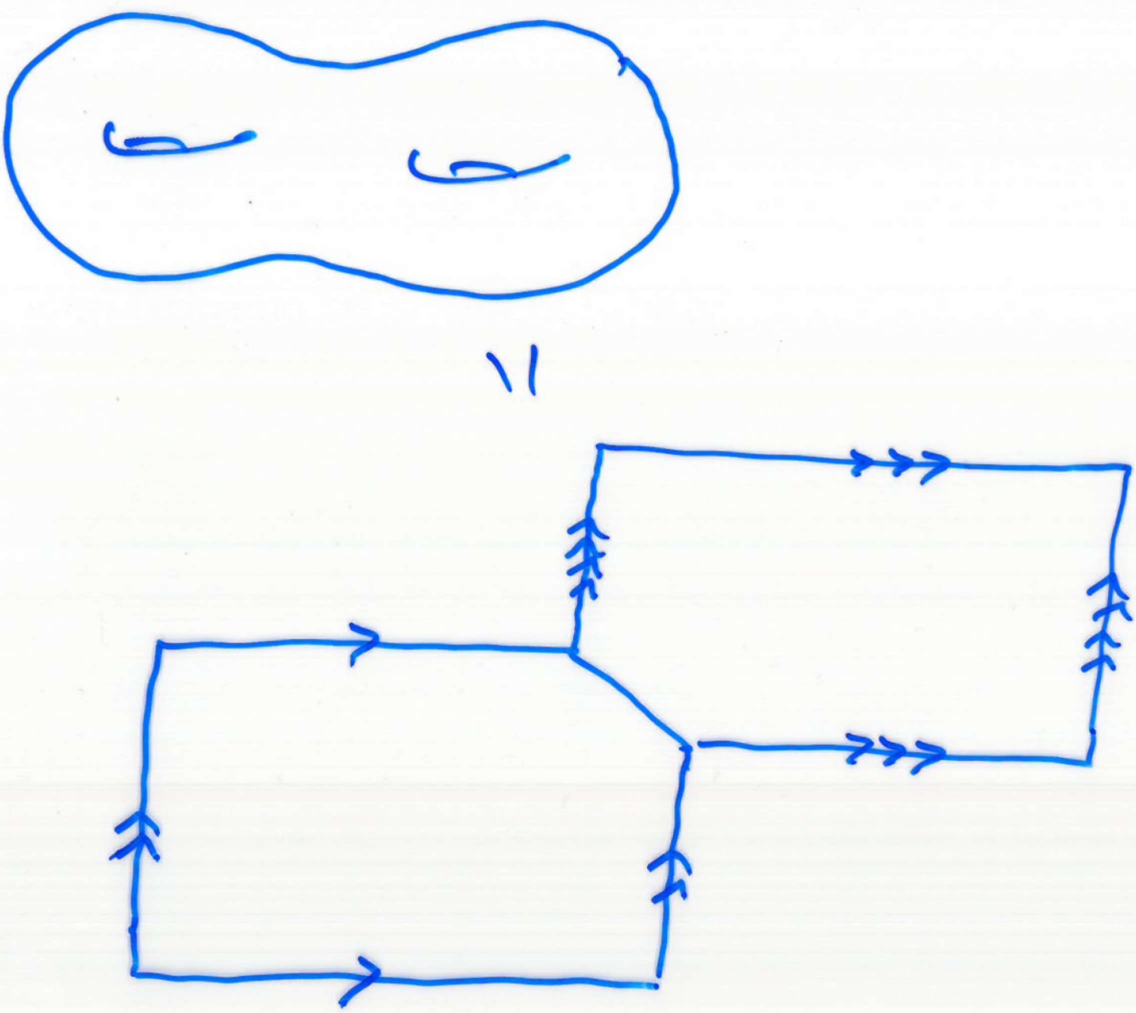
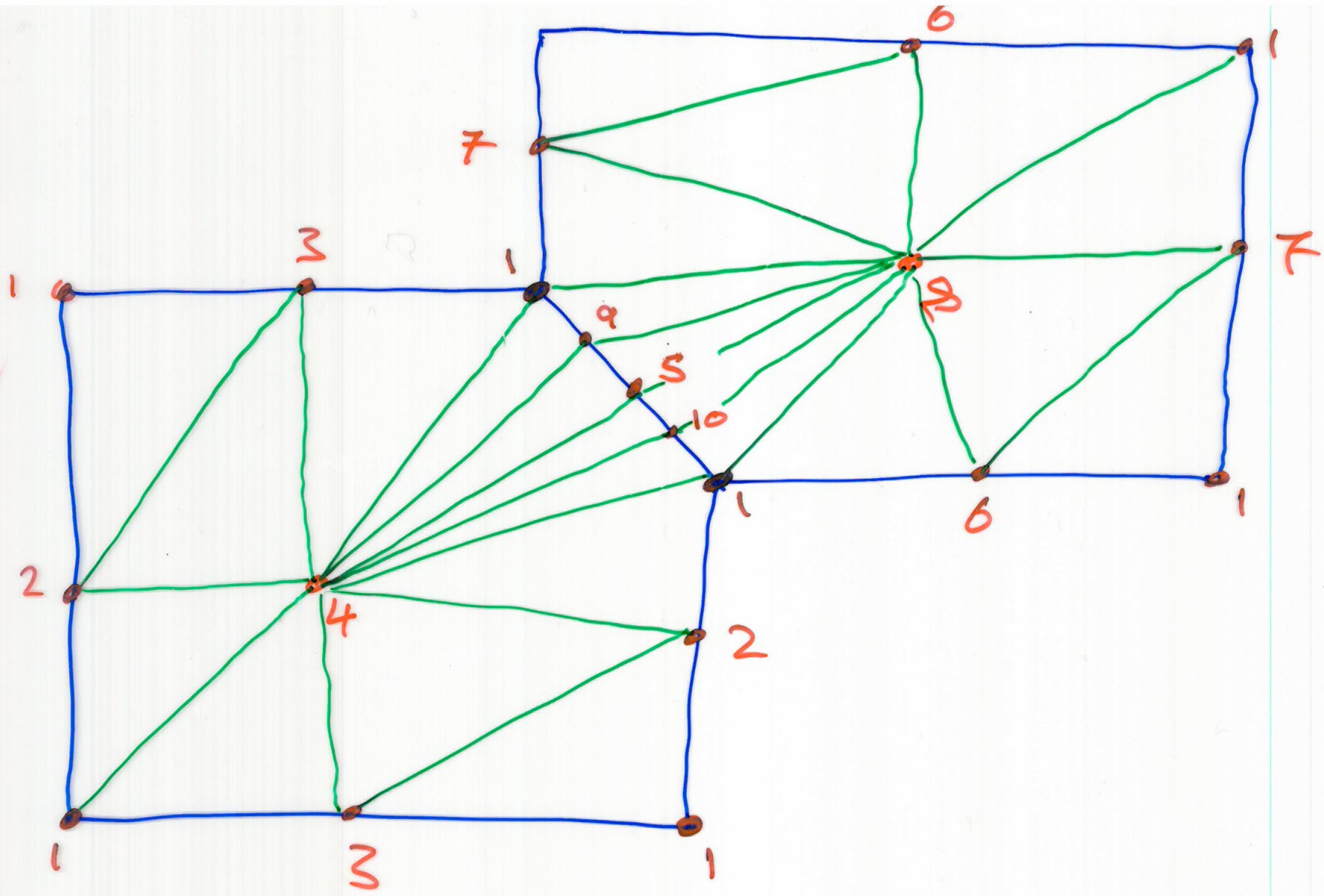
Example

Let's calculate the Euler characteristic of a "double torus".





$$\chi(\text{double torus}) = 10 - 36 + 24 = -2$$

Recall

$$D^n = \{ x \in \mathbb{E}^n : \|x\| \leq 1 \}$$

Brouwer's Theorem

For any continuous map

$$f : D^n \longrightarrow D^n$$

there exists at least one $x \in D^n$ such that $f(x) = x$.

Defn If $f(x) = x$ we say that x is a fixed point of the function f.

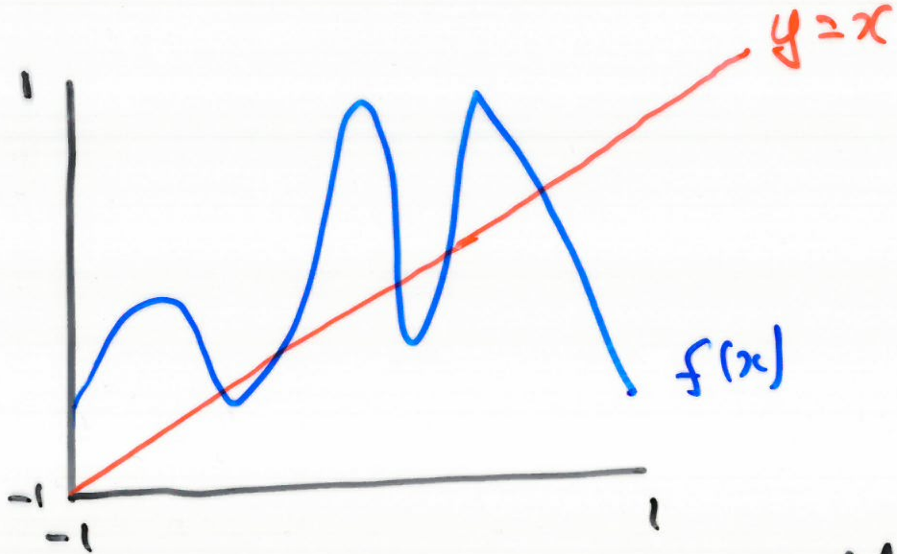
Case $n=1$

$D^1 = [-1, 1]$

We picture a map

$$f : [-1, 1] \to [-1, 1]$$

by its graph :



A fixed point is a point where the blue graph of $f(x)$ intersects the red line $y = x$.

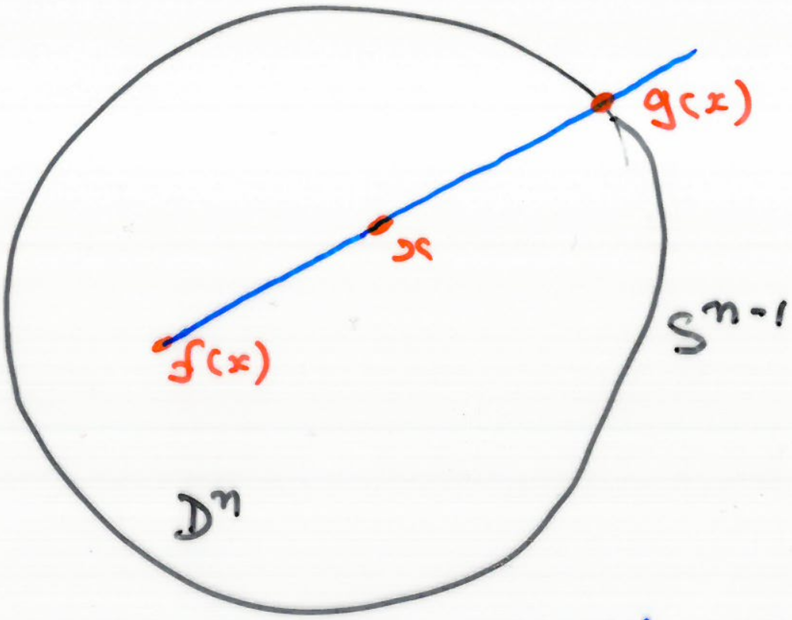
Proof of Brouwer's Theorem

Let $f: D^n \to D^n$ be continuous.

Suppose f has no fixed point.

Then we can define a continuous map

$$g: D^n \longrightarrow S^{n-1}, \quad x \longmapsto g(x)$$



where $g(x)$ is the point in S^{n-1} where the ray from $f(x)$ through x intersects S^{n-1}.

Note that $g(x)$ is continuous.

Let $h: S^{n-1} \longrightarrow D^n$, $x \longmapsto x$

Now

$$gh : S^{n-1} \longrightarrow S^{n-1}$$

is clearly the identity on S^{n-1}.

Now $hg: D^n \longrightarrow D^n$ is homotopic to the identity on D^n, via the homotopy

$$H(x,t) = x + t(g(x) - x)$$

$H(x,0) = x$

$H(x,1) = g(x) = gh(x)$

So

$$hg \simeq 1_{D^n},$$
$$gh \simeq 1_{S^{n-1}}.$$

Therefore D^n is homotopy equivalent to S^{n-1}.

Thus our Major theorem implies

$$\chi(D^n) = \chi(S^{n-1}).$$

But

$$D^n \simeq \{0\}$$

and so

$$\chi(D^n) = 1.$$

Last lecture:

$$\chi(S^{n-1}) = \begin{cases} 0 & n \text{ even} \\ 2 & n \text{ odd} \end{cases}$$

Contradiction!

Hence f has a fixed point.

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Theorem (Frobenius-Perron)

Let A be a real $n \times n$ matrix with entries $a_{ij} > 0$ for all i, j. Then A has a positive eigenvalue. Moreover, there is a corresponding eigenvector $v = (x_1, \dots, x_n)$ with $x_i \geq 0$.

Proof

Define $\sigma : \mathbb{R}^n \to \mathbb{R}, \ (x_1, \dots, x_n) = \sum_{i=1}^{n} x_i$.

$$\Delta^{n-1} = \left\{ (x_1, \dots, x_n) \in \mathbb{R}^n : \begin{array}{l} x_i \geq 0 \\ \text{and } \sum_{i=1}^{n} x_i = 1 \end{array} \right\}$$

Define

$$g : \Delta^{n-1} \longrightarrow \Delta^{n-1}, \quad x \longmapsto \frac{1}{\sigma(Ax)} Ax$$

Now g is continuous, and Δ^{n-1} is (homeomorphic to) D^{n-1}.

Brouwer's Theorem says that g has at least one fixed point.

$$x = g(x) = \frac{1}{\sigma(Ax)} Ax$$

So

$$Ax = \sigma(Ax) \cdot x$$

Therefore x is an eigenvector of g with eigenvalue

$\lambda = \sigma(Ax) \geq 0.$

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