$a_1 = 1, \quad a_{n+1} = a_n + \frac{1}{n}$

$\lim_{n \to \infty} a_n$ does not exist

even though $\lim_{n \to \infty} (a_{n+1} - a_n) = 0$

<u>Defn</u> A sequence of points $a_1, a_2, \ldots \in \mathbb{E}^d$ is said to be a <u>Cauchy sequence</u> if, for any $\varepsilon > 0$ there is an N such that

$$\| a_m - a_n \| < \varepsilon$$

for all $m, n > N$.

<u>Theorem</u> Any Cauchy sequence $a_1, a_2, \ldots$ in $\mathbb{E}^d$ has a limit $\lim_{n \to \infty} a_n$.

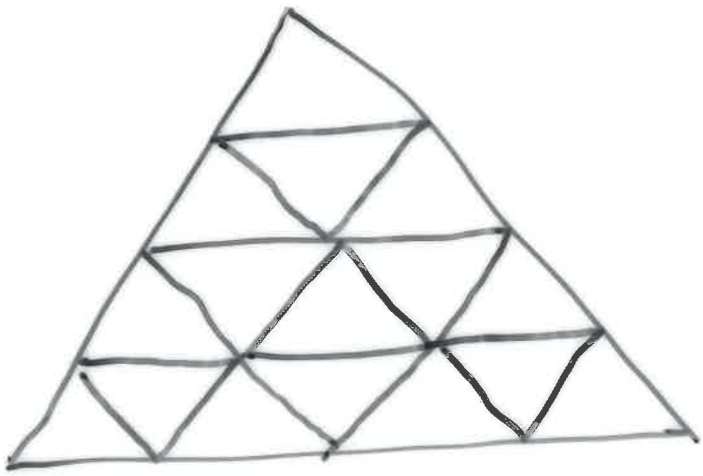
Last time we were constructing

$$f : [0,1] \longrightarrow \Delta$$

as

$$f(t) = \lim_{n\to\infty} f_n(t) .$$

where $f_n(t)$ was defined by subdividing Δ into small subtriangles



of side $\frac{1}{2^{n-1}}$. For fixed $t \in [0,1]$ the sequence $f_1(t), f_2(t), \ldots$ is a Cauchy sequence and hence converges by the above theorem.

If t is close to t' then $f(t)$ is close to $f(t')$ and thus, intuitively, we see that f is continuous.

It's not difficult to convert this intuition into a proof.

It remains to prove that $f(t)$ is surjective.

For this we'll use compactness of $[0,1]$ and "related results".

Some Theory

<u>Defn</u> Let X be a topological space. A subset $A \subseteq X$ is said to be <u>closed</u> if its complement $X \setminus A$ is open.

<u>Example</u> The subset $A = [0,1] \subseteq \mathbb{R}$ is a closed subset with respect to the usual topology on $\mathbb{R}$ since

$$\mathbb{R} \setminus [0,1] = (-\infty, 0) \cup (1, \infty)$$

is open in $\mathbb{R}$.

Example $(0,1]$ is neither open nor closed in $\mathbb{R}$.

Defn Let A be a subset of a topological space X. A point $p \in X$ is an accumulation point of A if every open subset U of X containing p also contains some point in $A \setminus \{p\}$.

Example Let $X = \mathbb{R}$ and $A = (0,1]$. Then every point in A is an accumulation point. So too is 0.

Example Let $X = \mathbb{R}$ with standard topology. Consider

$$A = \left\{ \frac{1}{n} \right\}_{n=1,2,3,\ldots}$$

In this example 0 is the only accumulation point.

Proposition A set A in a topological space X is closed if, and only if, it contains all its accumulation points.

Proof Suppose A is closed. Then $X \setminus A$ is open. Any point $x \in X \setminus A$ lies in the open set $X \setminus A$. So no point $x \in X \setminus A$ is an accumulation point. So any accumulation point must lie in A.

Conversely, suppose A contains all its accumulation points.

Let $x \in X \setminus A$. Since x is not an accumulation point we can find an open set $U_x \subseteq X \setminus A$.

So

$$X \setminus A = \bigcup_{x \in X \setminus A} U_x .$$

So $X \setminus A$ is open.

Hence A is closed. $\blacksquare$