Test: Wednesday 10 October.
5 questions taken from sections
1-7 of the problem sheet.

Continuity

A function $f(x,y)$ is continuous at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

exists, and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Example Consider

$$f(x,y) = \begin{cases} 3xy & (x,y) \neq (1,2) \\ 0 & (x,y) = (1,2) \end{cases}$$

At the point $(1,2)$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) =$$

$$= \lim_{(x,y)\to(1,2)} 3xy = 6$$

But

$$f(1,2) = 0.$$

Hence $f(x,y)$ is **not** continuous at $(1,2)$.

<u>Example</u> Consider

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Choose some constant m.

Suppose $x \to 0$. Then $y = mx \to 0$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = ?$$

$$\lim_{\substack{x\to 0\\ y=mx\to 0}} f(x,y) =$$

$$\lim_{x\to 0} \frac{x^2 - m^2x^2}{x^2 + m^2x^2}$$

$$= \lim_{x\to 0} \frac{1-m^2}{1+m^2}$$

$$= \frac{1-m^2}{1+m^2}.$$

This blue answer depends on m. Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ does <u>not</u> exist, it follows that $f(x,y)$ is <u>not</u> continuous at $(0,0)$.

Defn If a function $f(x,y)$ has continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at each point in a region S, then f is said to be continuously differentiable in the region.

Proposition If f is continuously differentiable in a region then f is continuous in the region, and f is differentiable in the region.

Example Consider

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

show that

$f_x(0,0)$ exists, and that

$f_y(0,0)$ exists, but that

$f(x,y)$ is not continuous at

$(0,0)$.

Sol^n $f_x(0,0)$

$$= \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h^2} = 0$$

$f_y(0,0)$

$$= \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h^2} = 0.$$

Consider $y = mx$, m constant.

$$\lim_{\substack{x \to 0 \\ y = mx \to 0}} \frac{xy}{x^2 + y^2} = \lim_{x \to 0} \frac{m x^2}{x^2 (1 + m^2)}$$

$$= \frac{m}{1 + m^2}.$$

This depends on m.

Hence $\lim\limits_{(x,y) \to (0,0)} f(x,y)$ does <u>not</u>

exist,

Hence $f(x,y)$ is <u>not</u> continuous

at $(0,0)$.

Partial Derivatives of composite Functions (Chain Rule)

If

$$u = F(x_1, x_2, \cdots, x_n)$$

where

$$x_1 = g_1(r_1, r_2, \cdots, r_p)$$

$$x_2 = g_2(r_1, r_2, \cdots, r_p)$$

$$\vdots$$

$$x_n = g_n(r_1, r_2, \cdots, r_p)$$

then

$$\frac{\partial u}{\partial r_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial r_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial r_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial r_i}.$$

Proof is easy and boring.

Example Consider

$$u = x^2 e^{yx}$$

where

$$x = t\cos(t)$$
$$y = t\sin(t).$$

Find

$$\frac{du}{dt} \text{ at } t = \frac{\pi}{2}.$$

Sol^n

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$= \left(2x e^{yx} + x^2 y e^{yx}\right)\left(\cos(t) - t\sin(t)\right)$$
$$+ \left(x^2 y\, e^{yx}\right)\left(\sin(t) + t\cos(t)\right)$$

Evaluate at $t = \frac{\pi}{2}$

etc.